



REGRESSION METHODS FOR HESITANT FUZZY PREFERENCE RELATIONS

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Abstract. In this paper, we develop two regression methods that transform hesitant fuzzy preference relations (HFPRs) into fuzzy preference relations (FPRs). On the basis of the complete consistency, reduced FPRs with the highest consistency levels can be derived from HFPRs. Compared with a straightforward method, this regression method is more efficient in the Matlab environment. Based on the weak consistency, another regression method is developed to transform HFPRs into reduced FPRs which satisfy the weak consistency. Two algorithms are proposed for the two regression methods, and some examples are provided to verify the practicality and superiority of the proposed methods.

Keywords: hesitant fuzzy preference relation (HFPR), fuzzy preference relation (FPR), complete consistency, weak consistency, consistency level.

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Introduction

Fuzzy preference relations (FPRs) are widely used in decision making, where consistency of FPRs is a major goal and interesting research topic (Herrera-Viedma *et al.* 2004, 2007; Jiang, Fan 2008; Tanino 1984, 1988; Wu *et al.* 2012; Wei *et al.* 2012; Stankevičienė, Mencaitė 2012; Baležentis *et al.* 2012). Recently, hesitant fuzzy sets (HFSs), originally introduced by Torra (2010), become a hot topic (Zhu *et al.* 2012a, b, 2013; Xu, Xia 2011). HFSs can consider the

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degrees of membership by a set of possible values. The motivation to propose HFSs is that when defining the membership of an element, the difficulty of establishing the membership degree is not a margin of error (as in intuitionistic fuzzy sets; Atanassov 1986), or some possibility distributions on the possible values (as in type 2 fuzzy sets; Zadeh 1975), but a set of possible values (Torra 2010).

With respect to the preference relations of HFSs, Xia and Xu (2013) defined hesitant fuzzy preference relations (HFPRs) and developed an approach to apply HFPRs to decision making. However, as a basic issue of HFPRs, the studies on consistency of HFPRs is not easy because the numbers of possible values in different hesitant fuzzy elements (HFEs) are often different. Since FPRs have been proven to be an effective tool used in decision making problems (Chiclana *et al.* 2001; Orlovsky 1978; Tanino 1984), we consider some techniques to transform HFPRs into FPRs based on their close relationship. Two regression methods are developed for the transformations based on the complete consistency and the weak consistency respectively.

The rest of this paper is organized as follows. Section 1 reviews some basic knowledge. In Section 2, we develop the regression methods, and illustrate their advantages with some examples. The final section ends the paper with some conclusions.

1. Preliminaries

This section introduces some concepts related to hesitant fuzzy sets (HFSs), fuzzy preference relations (FPRs), and hesitant fuzzy preference relations (HFPRs).

1.1. Hesitant fuzzy preference relations

Torra (2010) originally developed HFSs which cover arguments with a set of possible values.

Definition 1 (Torra 2010). Let X be a fixed set, a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of $[0,1]$.

To be easily understood, Xia and Xu (2011) expressed the HFS by a mathematical symbol:

$$E = \{ \langle x, h(x) \rangle \mid x \in X \}, \tag{1}$$

where $h(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set E . For convenience, Xia and Xu (2011) called h a hesitant fuzzy element (HFE).

For a HFE h , Xia and Xu (2011) developed some operations as follows:

- 1) $h^\lambda = \bigcup_{\gamma \in h} \{ \gamma^\lambda \}, \lambda > 0;$
- 2) $\lambda h = \bigcup_{\gamma \in h} \{ 1 - (1 - \gamma)^\lambda \}, \lambda > 0.$

FPRs (Orlovsky 1978) are an effective tool in decision making. The definition is as follows.

Definition 2 (Orlovsky 1978). A fuzzy preference relation (FPR) P on a set of objectives, X , is a fuzzy set on the product set $X \times X$, that is characterized by a membership function $\mu_p : X \times X \rightarrow [0,1]$.

When the cardinality of X is small, the fuzzy preference relation may be conveniently represented by a $n \times n$ matrix $P = (p_{ij})_{n \times n}$, where $p_{ij} = \mu_p(x_i, x_j)$. p_{ij} is interpreted as the preference degree of the objective x_i over x_j : $p_{ij} = 0.5$ indicates indifference between x_i and x_j , which can be represented by $x_i \sim x_j$; $p_{ij} = 1$ indicates that x_i is absolutely preferred x_j ; $p_{ij} > 0.5$ indicates that x_i is preferred to x_j , which can be represented by $x_i \succ x_j$. Generally, P is assumed to be additive reciprocal: $p_{ij} + p_{ji} = 1, i, j = 1, 2, \dots, n$.

On the basis of FPRs, Xia and Xu (2013) developed HFPRs which can be restated as follows.

Definition 3. Let $X = \{x_1, x_2, \dots, x_n\}$ be a fixed set, then a hesitant fuzzy preference relation (HFPR) H on X is presented by a matrix $H = (h_{ij})_{n \times n} \subset X \times X$, where $h_{ij} = \{\gamma_{ij}^l \mid l = 1, \dots, \#h_{ij}\}$ ($\#h_{ij}$ is the number of values in h_{ij}) is a HFE indicating all the possible preference degree(s) of the objective x_i over x_j . Moreover, h_{ij} should satisfy the following conditions:

$$\gamma_{ij}^{\sigma(l)} + \gamma_{ji}^{\sigma(\#h_{ij}-l+1)} = 1, \gamma_{ii} = \{0.5\}, \#h_{ij} = \#h_{ji}, i, j = 1, 2, \dots, n, \tag{2}$$

where: $\gamma_{ij}^{\sigma(l)}$ is the l th largest element in h_{ij} .

1.2. Consistency measures

The transitivity property is used to represent the idea that the preference degree obtained by directly comparing two objectives should be equal to or greater than the preference degree between those two objectives obtained using an indirect chain of objectives. This property is desirable to avoid contradictions reflected in preference relations. For the FPR $P = (p_{ij})_{n \times n}$, Tanino (1984) introduced an additive fuzzy transitivity property, or called the complete consistency:

$$p_{ij} + p_{jk} = p_{ik} + 0.5. \tag{3}$$

Tanino (1988) also introduced an additive fuzzy weak transitivity, or called the weak consistency: $p_{ij} \geq 0.5, p_{jk} \geq 0.5 \rightarrow p_{ik} \geq 0.5, i, j, k = 1, \dots, n$. It means that if x_i is preferred to x_j and x_j is preferred to x_k , then x_i should be preferred to x_k . This property verifies the condition that a logical and consistent person does not want to express his/her opinions with inconsistency, which guarantee the minimum requirement for consistency.

2. Regression methods for HFPRs

In this section, we develop two regression methods for HFPRs, which depend on the complete consistency and the weak consistency respectively.

2.1. A regression method for HFPRs based on the complete consistency

Herrera-Viedma *et al.* (2007) developed a method with error analysis to measure the consistency levels of FPRs. Motivated by this method, and based on the complete consistency and error analysis, we develop a regression method to transform HFPRs into FPRs.

Given a HFPR, represented by a matrix $H = (h_{ij})_{n \times n} \subset X \times X$, where $X = \{x_1, x_2, \dots, x_n\}$ is a fixed set of objectives. According to the definition of the complete consistency, the possible preference degrees over the paired objectives (i, k) represented by a HFE $h_{ik} (i \neq k)$ can be estimated using an intermediate objective $x_j (j \neq i, k)$:

$$h_{ik}^j = h_{ij} \tilde{+} h_{jk} \simeq 0.5, \tag{4}$$

where: h_{ik}^j can be called an estimated HFE, and the operations “ $\tilde{+}$ ” and “ \simeq ” are defined as follows.

Definition 4. Let h, h_1 and h_2 be three HFEs, and a be a real number, then we define

$$h_1 \tilde{+} h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2\}; \tag{5}$$

$$h \simeq a = \bigcup_{\gamma \in h} \{\gamma - a\}. \tag{6}$$

In order to use Eq. (4) to estimate h_{ik}^j , the objectives $x_i (i = 1, 2, \dots, n)$ should generally be classified into several sets defined as follows:

$$B = \{(i, k) | i, k \in \{1, 2, \dots, n\} \wedge (i \neq k)\}; \tag{7}$$

$$OV^B = \{(i, k) \in B\}; \tag{8}$$

$$KV^B = (OV^B)^c; \tag{9}$$

$$M_{ik}^j = \{j \neq i, k | (i, j), (j, k) \in KV^B\}, \tag{10}$$

where: B is a set of all paired objectives; OV^B is a set of paired objectives (i, k) ; KV^B is the complement set of OV^B satisfying $KV^B \cup OV^B = B$; M_{ik}^j is the set of the intermediate objectives $x_j (j \neq i, k)$.

Based on the discussions above and according to Eq.(4), we can get all the estimated HFE $h_{ik}^j (j = 1, 2, \dots, n; j \neq i, k)$. To select the optimal preference degree from $h_{ik}^j (j = 1, 2, \dots, n; j \neq i, k)$, we calculate an average estimated preference degree defined as follows:

$$h_{ik}^A = \frac{S_s \left(\bigcup_{j \in M_{ik}^j} h_{ik}^j \right)}{\sum_{j \in M_{ik}^j} (\# h_{ik}^j)}, \tag{11}$$

where: S_s is a function that indicates the summation of all elements in a set; $\# h_{ik}^j$ indicates the numbers of possible preference degrees in h_{ik}^j .

Comparing the possible values in the HFE h_{ik} and its average estimated preference degree h_{ik}^A , we define the error between them as follows.

Definition 5. For the HFE h_{ik} and its average estimated preference degree h_{ik}^A , the error between them is defined as:

$$\varepsilon h_{ik} = \frac{2}{3} (\cup_{\varepsilon_{ik} \in (h_{ik} - h_{ik}^A)} |\varepsilon_{ik}|), \tag{12}$$

where: the coefficient $2/3$ is used to make sure each value of the error belongs to the unit interval $[0,1]$.

If there exists a preference degree $h_{ik}^* (h_{ik}^* \in h_{ik})$ that corresponds to the minimum value of the error εh_{ik} satisfying:

$$\frac{2}{3} (h_{ik}^* - h_{ik}^A) = \min(\varepsilon h_{ik}), \tag{13}$$

then we should choose this preference degree as the optimal one. Following this principle and collecting h_{ik}^* for all $i, k = 1, 2, \dots, n; i \neq k$, we can transform H into a FPR $H^* = (h_{ik}^*)_{n \times n}$, which can be called a reduced FPR.

Further to measure the consistency level of H^* , we now give some definitions.

Definition 6. For the reduced FPR $H^* = (h_{ik}^*)_{n \times n}$, the consistency level of h_{ik}^* is defined as:

$$cl_{ik} = 1 - \min(\varepsilon h_{ik}). \tag{14}$$

With respect to one objective x_i , the consistency level is defined as follows.

Definition 7. For $H^* = (h_{ik}^*)_{n \times n}$, the consistency level of the objective x_i is defined as:

$$cl_i = \frac{\sum_{i \neq k, k=1}^n (cl_{ik} + cl_{ki})}{2(n-1)}. \tag{15}$$

So the consistency level of H^* can be further defined with respect to all the objectives.

Definition 8. For $H^* = (h_{ik}^*)_{n \times n}$, its consistency level is defined as:

$$cl_{H^*} = \frac{\sum_{i=1}^n cl_i}{n}. \tag{16}$$

Clearly, the bigger the value of $cl_{H^*} (cl_{H^*} \in [0,1])$, the higher the consistency level of H^* . Based on the analysis above, for a fixed set $X = \{x_1, x_2, \dots, x_n\}$, and a constructed HFPR $H = (h_{ij})_{n \times n}$, the algorithm that transforms H into H^* is shown in Algorithm I.

Algorithm I

Step 1. Randomly locate a HFE $h_{ik} (i \neq k)$, then calculate $h_{ik}^j (j = 1, 2, \dots, n; j \neq i, k)$ according to Eq. (4).

Step 2. Calculate the average estimated preference degree h_{ik}^A by Eq. (11), and then obtain h_{ik}^* by Eqs. (12) and (13).

Step 3. Repeat Steps 1 and 2 until all HFEs have been located, then turn to the next Step.

Step 4. Collecting h_{ik}^* for all $i, k = 1, 2, \dots, n (i \neq k)$, we can get the reduced FPR $H^* = (h_{ik}^*)_{n \times n}$.

Step 5. Calculate the consistency level of H^* according to Eqs. (14), (15) and (16).

Step 6. End.

Example 1. Assume a HFPR as follows:

$$H_1 = \begin{pmatrix} \{0.5\} & \{0.4, 0.5\} & \{0.6, 0.7\} & \{0.6\} \\ \{0.5, 0.6\} & \{0.5\} & \{0.8\} & \{0.4\} \\ \{0.3, 0.4\} & \{0.2\} & \{0.5\} & \{0.1, 0.2\} \\ \{0.4\} & \{0.6\} & \{0.8, 0.9\} & \{0.5\} \end{pmatrix}.$$

Step 1. Locate the HFE h_{12} , and according to Eq. (4), we have:

$$h_{12}^3 = h_{13} \tilde{+} h_{32} \simeq 0.5 = \{0.3, 0.4\}, h_{12}^4 = h_{14} \tilde{+} h_{42} \simeq 0.5 = \{0.7\}.$$

Step 2. According to Eq. (11), we have:

$$h_{12}^A = \frac{S_s \left(\sum_{j \in M_{12}^j} h_{12}^j \right)}{\sum_{j \in M_{12}^j} (\# h_{12}^j)} = \frac{(0.3 + 0.4) + 0.7}{2 + 1} = 0.467.$$

By Eqs. (12) and (13), we can get:

$$\varepsilon h_{12} = \frac{2}{3} \left(\bigcup_{\varepsilon_{12} \in (h_{12} \simeq h_{12}^A)} |\varepsilon_{12}| \right) = \{0.044, 0.022\}, \min(\varepsilon h_{12}) = 0.022 = \frac{2}{3} |0.5 - 0.467|.$$

Thus, $h_{12}^* = 0.5$.

Step 3. Repeat Steps 1 and 2, we have:

$$h_{13}^* = 0.7, \min(\varepsilon h_{13}) = 0.100; h_{14}^* = 0.6, \min(\varepsilon h_{14}) = 0.189;$$

$$h_{21}^* = 0.5, \min(\varepsilon h_{21}) = 0.022; h_{23}^* = 0.8, \min(\varepsilon h_{23}) = 0.056;$$

$$h_{24}^* = 0.4, \min(\varepsilon h_{24}) = 0.100; h_{31}^* = 0.3, \min(\varepsilon h_{31}) = 0.067;$$

$$h_{32}^* = 0.2, \min(\varepsilon h_{32}) = 0.056; h_{34}^* = 0.2, \min(\varepsilon h_{34}) = 0.089;$$

$$h_{41}^* = 0.4, \min(\varepsilon h_{41}) = 0.189; h_{42}^* = 0.6, \min(\varepsilon h_{42}) = 0.100;$$

$$h_{43}^* = 0.8, \min(\varepsilon h_{43}) = 0.089.$$

Step 4. Collecting h_{ik}^* for all $i, k = 1, 2, \dots, n (i \neq k)$, we can get the reduced FPR H_1^* as:

$$H_1^* = \begin{pmatrix} 0.5 & 0.5 & 0.7 & 0.6 \\ 0.5 & 0.5 & 0.8 & 0.4 \\ 0.3 & 0.2 & 0.5 & 0.2 \\ 0.4 & 0.6 & 0.8 & 0.5 \end{pmatrix}.$$

Step 5. According to Eq. (14), we have:

$$cl_{12} = 0.978 ; cl_{21} = 0.978 ; cl_{13} = 0.900 ; cl_{31} = 0.933 ; cl_{14} = 0.811 ; cl_{41} = 0.811 ;$$

$$cl_{23} = 0.944 ; cl_{32} = 0.944 ; cl_{24} = 0.900 ; cl_{42} = 0.900 ; cl_{34} = 0.911 ; cl_{43} = 0.911.$$

According to Eq. (15), we can get:

$$cl_1 = \frac{(cl_{12} + cl_{21}) + (cl_{13} + cl_{31}) + (cl_{14} + cl_{41})}{6} = 0.902;$$

$$cl_2 = \frac{(cl_{21} + cl_{12}) + (cl_{23} + cl_{32}) + (cl_{24} + cl_{42})}{6} = 0.941;$$

$$cl_3 = \frac{(cl_{31} + cl_{13}) + (cl_{32} + cl_{23}) + (cl_{34} + cl_{43})}{6} = 0.924;$$

$$cl_4 = \frac{(cl_{41} + cl_{14}) + (cl_{42} + cl_{24}) + (cl_{43} + cl_{34})}{6} = 0.874.$$

Furthermore, by Eq. (16), the consistency level of H_1^* is:

$$cl_{H_1^*} = \frac{cl_1 + cl_2 + cl_3 + cl_4}{4} = 0.91.$$

Step 6. End.

For the HFPR $H = (h_{ij})_{n \times n}$, since each preference degree in h_{ij} is a possible value, H can be directly separated into all possible FPRs. Then based on some existing consistency measure methods, the FPR with the highest consistency level can be found out. In order to compare this straightforward method with our method, we give the following example.

Example 2. Based on the same HFPR H_1 , we can generate eight possible FPRs from H_1 denoted as follows:

$$P_1^{H_1} = \begin{pmatrix} 0.5 & 0.5 & 0.6 & 0.6 \\ 0.5 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.1 \\ 0.4 & 0.6 & 0.9 & 0.5 \end{pmatrix}; P_2^{H_1} = \begin{pmatrix} 0.5 & 0.5 & 0.7 & 0.6 \\ 0.5 & 0.5 & 0.8 & 0.4 \\ 0.3 & 0.2 & 0.5 & 0.1 \\ 0.4 & 0.6 & 0.9 & 0.5 \end{pmatrix}; P_3^{H_1} = \begin{pmatrix} 0.5 & 0.5 & 0.6 & 0.6 \\ 0.5 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.2 \\ 0.4 & 0.6 & 0.8 & 0.5 \end{pmatrix};$$

$$P_4^{H_1} = \begin{pmatrix} 0.5 & 0.5 & 0.7 & 0.6 \\ 0.5 & 0.5 & 0.8 & 0.4 \\ 0.3 & 0.2 & 0.5 & 0.2 \\ 0.4 & 0.6 & 0.8 & 0.5 \end{pmatrix}; P_5^{H_1} = \begin{pmatrix} 0.5 & 0.4 & 0.6 & 0.6 \\ 0.6 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.1 \\ 0.4 & 0.6 & 0.9 & 0.5 \end{pmatrix}; P_6^{H_1} = \begin{pmatrix} 0.5 & 0.4 & 0.7 & 0.6 \\ 0.6 & 0.5 & 0.8 & 0.4 \\ 0.3 & 0.2 & 0.5 & 0.1 \\ 0.4 & 0.6 & 0.9 & 0.5 \end{pmatrix};$$

$$P_7^{H_1} = \begin{pmatrix} 0.5 & 0.4 & 0.6 & 0.6 \\ 0.6 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.2 \\ 0.4 & 0.6 & 0.8 & 0.5 \end{pmatrix}; P_8^{H_1} = \begin{pmatrix} 0.5 & 0.4 & 0.7 & 0.6 \\ 0.6 & 0.5 & 0.8 & 0.4 \\ 0.3 & 0.2 & 0.5 & 0.2 \\ 0.4 & 0.6 & 0.8 & 0.5 \end{pmatrix}.$$

According to the consistency measure of FPRs introduced by Herrera-Viedma *et al.* (2007), we can get the consistency levels of $P_i^{H_1}$ ($i=1,2,\dots,8$) denoted by:

$$cl_{P_1^{H_1}} = 89.63\%; cl_{P_2^{H_1}} = 91.76\%; cl_{P_3^{H_1}} = 90.56\%; cl_{P_4^{H_1}} = 92.69\%;$$

$$cl_{P_5^{H_1}} = 88.52\%; cl_{P_6^{H_1}} = 90.65\%; cl_{P_7^{H_1}} = 89.44\%; cl_{P_8^{H_1}} = 91.57\%.$$

Since:

$$cl_{P_4^{H_1}} > cl_{P_2^{H_1}} > cl_{P_8^{H_1}} > cl_{P_6^{H_1}} > cl_{P_3^{H_1}} > cl_{P_1^{H_1}} > cl_{P_7^{H_1}} > cl_{P_5^{H_1}},$$

$P_4^{H_1}$ is with the highest consistency level.

Obviously, $P_4^{H_1} = H_1^*$. So the results are the same by the regression method and the straightforward method. Considering the efficiency, the number of operations of our regression method and the straightforward method are $2n(n-1)+n+1$ and $m(n(n-1)+n+1)$ (m is the number of all possible FPRs separated from a HFPR), respectively. Since $m \geq 2$ (at least two FPRs can be separated from a HFPR), we have $m(n(n-1)+n+1) > 2n(n-1)+n+1$. So our method is simpler. Moreover, the bigger the value m , the simpler the regression method.

2.2. A regression method for HFPRs based on the weak consistency

For the decision making problems in practical applications, the complete consistency is sometimes not necessary due to the complicated environment and the cognitive diversity of humans. But, the weak consistency is essential because a contradictory HFPR doesn't make sense. On the basis of the weak consistency, we now develop another regression method to get reduced FPRs satisfying the weak consistency. In what follows, we begin with some necessary definitions and discussions.

Definition 9. Assume a HFPR, $H = (h_{ij})_{n \times n} = \bigcup_{\gamma_{ij} \in h_{ij}} \{\gamma_{ij}\}_{n \times n}$, its hesitant preference degree (HPD) is defined as:

$$m_{ij} = \bigcup_{s_{ij} \in m_{ij}^H} \{s_{ij}\}, \tag{17}$$

where: s_{ij} is called a hesitant preference element (HPE), satisfying:

$$s_{ij} = \begin{cases} 1, & \text{if } 0.5 < \gamma_{ij} \leq 1 \\ 0, & \text{if } 0 \leq \gamma_{ij} < 0.5 \end{cases} \tag{18}$$

Then $M = (m_{ij})_{n \times n}$ is called a hesitant preference relation (HPR).

According to graph theory (Bondy, Murty 1976), the relationship included in the HPR can be described by a directed graph which can be called a hesitant fuzzy preference graph. In such a graph, each node stands for an objective, and each directed edge stands for a preference relation. If $m_{ij} = 1$, then there is a directed edge from a node i to a node j , which represents that the objective i is superior to the objective j .

Example 3. Assume a fixed set $X = \{x_1, x_2, x_3, x_4\}$, and two constructed HFPRs as

$$H_2 = \begin{pmatrix} \{0.5\} & \{0.6\} & \{0.6,0.7\} & \{0.6\} \\ \{0.4\} & \{0.5\} & \{0.8\} & \{0.4\} \\ \{0.3,0.4\} & \{0.2\} & \{0.5\} & \{0.2\} \\ \{0.4\} & \{0.6\} & \{0.8\} & \{0.5\} \end{pmatrix}; H_3 = \begin{pmatrix} \{0.5\} & \{0.4,0.6\} & \{0.6\} & \{0.4,0.6\} \\ \{0.4,0.6\} & \{0.5\} & \{0.8\} & \{0.4\} \\ \{0.4\} & \{0.2\} & \{0.5\} & \{0.2,0.3\} \\ \{0.4,0.6\} & \{0.6\} & \{0.7,0.8\} & \{0.5\} \end{pmatrix}.$$

According to Definition 9, we get:

$$M_{H_2} = \begin{pmatrix} \{0\} & \{1\} & \{1\} & \{1\} \\ \{0\} & \{0\} & \{1\} & \{0\} \\ \{0\} & \{0\} & \{0\} & \{0\} \\ \{0\} & \{1\} & \{1\} & \{0\} \end{pmatrix}; M_{H_3} = \begin{pmatrix} \{0\} & \{1,0\} & \{1\} & \{1,0\} \\ \{0,1\} & \{0\} & \{1\} & \{0\} \\ \{0\} & \{0\} & \{0\} & \{0\} \\ \{0,1\} & \{1\} & \{1\} & \{0\} \end{pmatrix}.$$

The hesitant fuzzy preference graphs of M_{H_2} and M_{H_3} are shown in Figures 1 and 2 respectively.

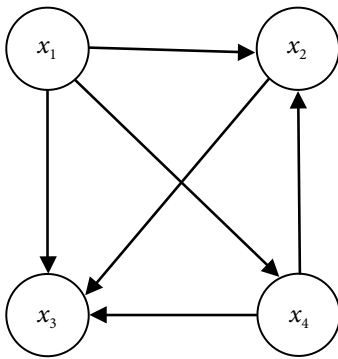


Fig. 1. Hesitant fuzzy preference graph of M_{H_2}

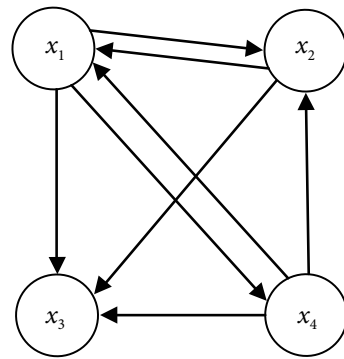


Fig. 2. Hesitant fuzzy preference graph of M_{H_3}

With respect to a HFPR, if there is no circular triad in a hesitant fuzzy preference graph, it means that a circular relation of objectives does not exist. So the HFPR satisfies the weak consistency, such as in Fig. 1. However, in Fig. 2, we can see that the objectives x_1 and x_4 are connected by two opposite directed edges. In such a case, we can get a circular triad of objectives as $x_1 \rightarrow x_4 \rightarrow x_2 \rightarrow x_1$ in Fig. 2. Thus the corresponding HFPR, H_3 , does not satisfy the weak consistency. Therefore, the circular triad can be used to test the weak consistency of HFPRs, which is defined as follows.

Definition 10. Let $M = (m_{ij})_{n \times n}$ be the HPR of a HFPR $H = (h_{ij})_{n \times n}$, where $m_{ij} = \bigcup_{s_{ij} \in m_{ij}} \{s_{ij}\}$, then:

$$C_{ijk} = \bigcup_{c_{ijk} \in C_{ijk}} \{c_{ijk}\} = \bigcup_{s_{ij} \in m_{ij}, s_{jk} \in m_{jk}, s_{ki} \in m_{ki}} \{s_{ij} + s_{jk} + s_{ki}\} \quad (i, j, k \in \{1, 2, \dots, n\}; i \neq j \neq k), \quad (19)$$

is called a circular triad power, and c_{ijk} is called a circular triad power element.

Theorem 1. For a HFPR, $H = (h_{ij})_{n \times n}$, we can get its HPR $M = (m_{ij})_{n \times n}$, and the circular triad power $C_{ijk} = \bigcup_{c_{ijk} \in C_{ijk}} \{c_{ijk}\}$. If and only if there exists $c_{ijk} = 3$, then H does not satisfy the weak consistency.

Proof. If $c_{ijk} = 3$ exists, then there exists one circular triad indicating a relation of objectives as $x_i \succ x_j \succ x_k \succ x_i$. According to the definition of the weak consistency, H does not satisfy the weak consistency; if H does not satisfy the weak consistency, then there exists a circular triad of objectives as $x_i \rightarrow x_j \rightarrow x_k \rightarrow x_i$. According to Definition 10, there exists $c_{ijk} = 3$, which complete the proof.

Jiang and Fan (2008) gave a definition of a reachability matrix used to test the weak consistency of FPRs. Motivated by this idea, and based on Theorem 1, we now develop a hesitant reachability matrix (HRM) to identify the weak consistency of HFPRs.

Definition 11. Let $M = (m_{ij})_{n \times n}$ be the HPR of a HFPR $H = (h_{ij})_{n \times n}$, then we call $M^{(k)} = (m_{ij}^{(k)})_{n \times n}$ the k th power of M , where the (i, j) entry, denoted by $m_{ij}^{(k)}$, is the number of different directed edges of the length k from the node i to the node j .

Furthermore, we define the HRM as follows.

Definition 12. Let $M = (m_{ij})_{n \times n}$ be the HPR of a HFPR $H = (h_{ij})_{n \times n}$, $M^{(3)} = (m_{ij}^{(3)})_{n \times n}$ be the third power of M , then we call the matrix $R = (r_{ij})_{n \times n}$ the hesitant reachability matrix (HRM), where $R = M^{(3)}$.

Theorem 2. For a HFPR, $H = (h_{ij})_{n \times n}$, if all diagonal elements are zero in its hesitant reachability matrix $R = (r_{ij})_{n \times n}$, then H satisfies the weak consistency.

Proof. For the HFPR, $H = (h_{ij})_{n \times n}$, if all the diagonal elements are zero in its hesitant reachability matrix $R = (r_{ij})_{n \times n}$, i.e. $M^{(3)} = (m_{ii}^{(3)})_{n \times n} = 0$, then according to Theorem 1, we know that there is no circular triad in H . So H satisfies the weak consistency, which completes the proof.

According to Definition 12, and based on the two HFPRs H_2 and H_3 in Example 3, we can get two HRMs as follows:

$$R_{H_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; R_{H_3} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$

According to Theorem 2, H_2 satisfies the weak consistency, but H_3 does not.

Based on the discussions above, we now give a step by step procedure to obtain reduced FPRs satisfying the weak consistency shown in Algorithm II.

Algorithm II

Given a HFPR, $H^p = (h_{ij}^p)_{n \times n}$ ($p=0$; H^p is the p th power of H indicating the number of being modified).

Step 1. According to Definitions 9 and 12, we can get its HPR $M^p = (m_{ij})_{n \times n}$ and HRM $R^p = (r_{ij})_{n \times n}$, respectively.

Step 2. According to Theorem 2, if H^p satisfies the weak consistency, turn to Step 5; otherwise, turn to Step 3.

Step 3. For the HPR $M^p = (m_{ij})_{n \times n}$, where $m_{ij} = \bigcup_{s_{ij} \in m_{ij}} \{s_{ij}\}$, locate a pair of HPDs (m_{ij}, m_{ji}) , satisfying $m_{ij} = \{0,1\}$, $m_{ji} = \{1,0\}$.

According to Eq. (19), find out a $c_{ijk} = 3$, remove the pair of HPEs (s_{ij}, s_{ji}) satisfying $s_{ij} + s_{ji} = 1$, and remove their corresponding preference degrees in the pair of HFES (h_{ij}, h_{ji}) in H .

Step 4. Let $p = p + 1$, construct a modified HFPR as H^{p+1} , turn to Step 1.

Step 5. Divide H^p into all possible reduced FPRs.

Step 6. End.

Example 4. Continued with H_3 in Example 3, let H_3 be $H_3^p (p=0)$. Since H_3^p does not satisfy the weak consistency, we turn to Step 3.

Step 3. Locate (m_{14}, m_{41}) , satisfying $m_{14} = \{0,1\}$ and $m_{41} = \{1,0\}$. According to Eq. (19), we can find $c_{142} = 3$. Remove the pair of HPEs $s_{14} = \{0\}$ and $s_{41} = \{1\}$, and remove their corresponding preference degrees $\gamma_{14} = 0.4$, $\gamma_{41} = 0.6$ in the pair of HFES (h_{14}, h_{41}) .

Step 4. Let $p = p + 1$, construct a modified HFPR H_3^1 as follows:

$$H_3^1 = \begin{pmatrix} \{0.5\} & \{0.4, 0.6\} & \{0.6\} & \{0.4\} \\ \{0.4, 0.6\} & \{0.5\} & \{0.8\} & \{0.4\} \\ \{0.4\} & \{0.2\} & \{0.5\} & \{0.2, 0.3\} \\ \{0.6\} & \{0.6\} & \{0.7, 0.8\} & \{0.5\} \end{pmatrix}.$$

Then turn to Step 1.

Step 1. According to Definitions 9 and 12, and the modified HHPR H_3^1 , we can get:

$$M_{H_3^1} = \begin{pmatrix} \{0\} & \{1,0\} & \{1\} & \{0\} \\ \{0,1\} & \{0\} & \{1\} & \{0\} \\ \{0\} & \{0\} & \{0\} & \{0\} \\ \{1\} & \{1\} & \{1\} & \{0\} \end{pmatrix}; R_{H_3^1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Step 2. According to Theorem 2, H_3^1 satisfies the weak consistency, turn to Step 5.

Step 5. Divide H_3^1 into the following possible reduced FPRs satisfying the weak consistency:

$$P_1^{H_3^1} = \begin{pmatrix} 0.5 & 0.4 & 0.6 & 0.4 \\ 0.6 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.2 \\ 0.6 & 0.6 & 0.8 & 0.5 \end{pmatrix}; P_2^{H_3^1} = \begin{pmatrix} 0.5 & 0.4 & 0.6 & 0.4 \\ 0.6 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.7 & 0.5 \end{pmatrix};$$

$$P_3^{H_3^1} = \begin{pmatrix} 0.5 & 0.6 & 0.6 & 0.4 \\ 0.4 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.2 \\ 0.6 & 0.6 & 0.8 & 0.5 \end{pmatrix}; P_4^{H_3^1} = \begin{pmatrix} 0.5 & 0.6 & 0.6 & 0.4 \\ 0.4 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.7 & 0.5 \end{pmatrix}.$$

Step 6. End.

In practical applications, Algorithms I and II can be combined to obtain reduced FPRs from HFPRs, where the obtained reduced FPR can not only satisfy the weak consistency but also have the highest confidence level.

For example, we replace Step 5 in Example 4 by Algorithm I. Then we can obtain a reduced FPR, denoted by H_2^* , with the highest consistency level 95.56% :

$$H_2^* = \begin{pmatrix} 0.5 & 0.4 & 0.6 & 0.4 \\ 0.6 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.2 \\ 0.6 & 0.6 & 0.8 & 0.5 \end{pmatrix}.$$

Conclusions

For a hesitant fuzzy preference relation (HFPR), it is not easy to deal with its consistency due to different numbers of possible values in hesitant fuzzy elements (HFEs). In this paper, we have developed two regression methods to transform HFPRs into reduced fuzzy preference relations (FPRs). Based on the complete consistency, we use error analysis to select the optimal preference degree for each paired objectives in HFPRs to produce reduced FPRs. The step by step procedure of this regression method is shown in Algorithm I. On the basis of the weak consistency, we have defined a hesitant preference relation (HPR) and a circular triad power to find circular triads of objectives in HFPRs. Then we have given Theorems 1 and 2 to identify the weak consistency of HFPRs. With these definitions and methods, we have given Algorithm II to transform HFPRs into FPRs that satisfy the weak consistency.

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