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A GLOBAL CHARACTERISTIC OF G-LIMIT OPERATORS FOR QUASILINEAR POTENTIAL ELLIPTIC SYSTEMS[1]

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ABSTRACT

The paper considers a family of quasilinear potential elliptic systems and uses the fact that all G-limit operators for this family can be characterized by means of gradients of convex functions F (locally with respect to the spatial co-ordinates). It is shown that all these functions F must satisfy an inequality expressed in terms of functions F and their conjugate functions.

Key words: elliptic systems, convex functions, G-limit operators.

1. INTRODUCTION

In this paper, we give a characteristic of G-limit operators for simple quasilinear potential elliptic systems of the kind

$$\begin{aligned} &\operatorname{div}\Big[\sum_{s=1}^{N} \sigma_s(x) F_s'(\nabla \overline{u}(x) + g(x)) - f(x)\Big] = 0 \ \text{ in } \Omega, \\ &\overline{u} = (u_1, \dots, u_m) \in H_0^1(\Omega; \mathbf{R}^m), \end{aligned} \tag{1.1}$$

depending on the functional parameter σ

$$\begin{aligned} &\sigma \in S = \{\sigma \in L_\infty(\mathbf{R}^n; \mathbf{R}^N) \mid \sigma = (\sigma_1, \dots, \sigma_N), \\ &\ \sigma_j(x) = 0 \text{ or } 1, \ j = 1, \dots, N; \\ &\sigma_1(x) + \dots + \sigma_N(x) = 1 \ \text{ a.e. } x \in \mathbf{R}^n\}. \end{aligned}$$

Here F_s, $s = 1, \dots, N$, are given strictly convex smooth functions with quadratic growth, $F_s'(\cdot)$ is the gradient of $F_s(\cdot)$, $s = 1, \dots, N$.

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It is known, see for instance, Raitums [3], that all G-limit operators for the family (1.1) can be fully characterized by the set $\mathcal{A}$ of functions

$$\mathcal{A} = \{F_\sigma \in C^1(\mathbf{R}^{nm}) \mid \sigma \in S\}$$

where for a chosen $\sigma \in S$

$$F_\sigma(z) = \inf_{v \in H^\#} \int_K \sum_{s=1}^{N} \sigma_s(x) F_s(v(x) + z) dx, \quad \forall z \in \mathbf{R}^{nm}.$$

Here $K \subset \mathbf{R}^n$ is the unit cube, $K = (0,1)^n$, and

$$H^\# = \{v \in L_2(K; \mathbf{R}^{nm}) \mid v = \nabla \overline{u},\ \overline{u} \in H^1_{loc}(\mathbf{R}^n, \mathbf{R}^m),\ \overline{u} \text{ is } K-\text{periodic}\}.$$

One of the main features of the nonlinear (the functions $F_s'(\cdot)$ are not affine) case is that the properties of $F_\sigma(\cdot)$ at a point z depend on the behaviour of F_s on the whole $\mathbf{R}^{nm}$, i.e. these properties are not local.

We are interested in the question of the existence of a larger set $\hat{\mathcal{A}} \subset C^1(\mathbf{R}^{nm})$ of strictly convex functions $\hat{F}$ such that if for some $z' \in \mathbf{R}^{nm}$

$$\hat{F}'(z') = z'',$$

then there exists a sequence $\{F_{\sigma^k}\} \subset \mathcal{A}$ such that

$$F'_{\sigma^k}(z') \to z'' \text{ as } k \to \infty.$$

If such a set $\hat{\mathcal{A}}$ (with some additional properties) exists then by standard methods of the theory of G-convergence and monotone operators one can easy show that for every solution $\overline{u}^0$ of the system

$$\operatorname{div}\Big(\sum_{s=1}^{s_0} \chi_{\Omega_s}(x) \hat{F}_s'(\nabla \overline{u}(x) + g(x)) - f(x)\Big) = 0 \text{ in } \Omega,$$
$$\overline{u} \in H^1_0(\Omega; \mathbf{R}^m),$$

there exists a sequence $\{\sigma^k\} \subset S$ such that the sequence $\{\overline{u}^k\}$ of solutions of the system (1.1) with $\sigma = \sigma^k$, $k = 1, 2, \ldots$, respectively converges weakly in $H^1_0(\Omega; \mathbf{R}^m)$ to $\overline{u}^0$ as $k \to \infty$. Here $\{\Omega_s\}$ is a partition of Ω by means of pairwise disjoint measurable sets Ω_s, χ_{Ω_s} is the characteristic function of Ω_s and $\hat{F}_s \in \hat{\mathcal{A}}$, $s = 1, \ldots, s_0$.

In this sense the passage from $\{F_1; \ldots; F_N\}$ to $\mathcal{A}$ and further to $\hat{\mathcal{A}}$ preserves the weak closure of the set of all feasible solutions of the family (1.1).

In what follows (Section 3) we shall show that if the functions F_s, $s = 1, \ldots, N$, satisfy some hypotheses (see Section 2) then there exists such a set $\hat{\mathcal{A}}$ and its main functional characteristic is

$$\hat{F}(z') + \hat{F}^*(z'') \geq Q\mathcal{F}(z', z'') \quad \forall z', z'' \in \mathbf{R}^{nm},$$

where $Q\mathcal{F}$ is some $(\text{curl}, \text{div})^m$ – quasiconvex function and by $\hat{F}^*$ is denoted the conjugate to $\hat{F}$ function.

2. PRELIMINARIES

Let $n \geq 2$, $m \geq 1$, $N \geq 2$ be integers and let the functions $F_s : \mathbf{R}^{nm} \to \mathbf{R}$, $s = 1, \ldots, N$, satisfy the following hypotheses.

H1. $F_s, s = 1, \ldots, N$, belong to $C^1(\mathbf{R}^{nm})$ and are strictly convex.

H2. There exist positive constants ν, μ such that for all $z, \xi \in \mathbf{R}^{nm}$

$$|F_s'(z) - F_s'(\xi)| \leq \mu |z - \xi|,$$
$$\langle F_s'(z + \xi) - F_s'(z), \quad \xi \rangle \geq \nu |\xi|^2, \quad s = 1, \ldots, N.$$

H3. $F_s(0) = 0$, $F_s'(0) = 0$, $s = 1, \ldots, N$.

For a given function $F : \mathbf{R}^{nm} \to \mathbf{R}$ by F^* we shall denote its conjugate function, i.e.

$$F^*(z'') = \sup_{z' \in \mathbf{R}^{nm}} [\langle z', z'' \rangle - F(z')].$$

From **H1** – **H3** it follows immediately that F_s and F_s^* are nonnegative and that the conjugate functions F_s^* have analogous to **H1** – **H3** properties (with different constants ν', μ' instead of ν, μ).

Let $K \subset \mathbf{R}^n$ be a unit cube, $K = (0,1)^n$, and let

$$H^{\#} = \{v \in L_2(K; \mathbf{R}^{nm}) \mid v = \nabla \overline{u},\ \overline{u} \in H^1_{loc}(\mathbf{R}^n; \mathbf{R}^m),\ \overline{u} \text{ is } K - \text{periodic}\},$$
$$N^{\#} = \{\eta \in L_2(K; \mathbf{R}^{nm}) \mid \eta = (\eta_1, \ldots, \eta_m),\ \eta_j = \text{div}\,\mathcal{U}_j,$$
$$\mathcal{U}_j \in H^1_{loc}(\mathbf{R}^n; \mathbf{R}^{n \times n}) \text{ is a skew-symmetric}$$
$$\text{and } K - \text{periodic } n \times n - \text{matrix},\ j = 1, \ldots, m\}.$$

We denote by $E(\nu', \mu')$ the set of all functions $F : \mathbf{R}^{nm} \to \mathbf{R}$ which satisfy the hypotheses **H1** – **H3** with ν', μ' instead of ν, μ. Let

$$\mathcal{A} = \{F_\sigma \in C(\mathbf{R}^{nm}) \mid F_\sigma(z) = \inf_{v \in H^{\#}} \int_K \sum_{s=1}^N \sigma_s(x) F_s(v(x) + z) dx,\ \sigma \in S\},$$

let the function $Q\mathcal{F} \in C(\mathbf{R}^{nm} \times \mathbf{R}^{nm})$ be defined as

$$Q\mathcal{F}(z', z'') = \inf_{\sigma \in S} \inf_{v \in H^{\#}} \inf_{\eta \in N^{\#}} \int_K \sum_{s=1}^N \sigma_s(x) \Big(F_s(v(x) + z') + F_s^*(\eta(x) + z'')\Big) dx \tag{2.1}$$

and let

$$\hat{\mathcal{A}} = \{\hat{F} \in E(\nu, \frac{\mu^2}{\nu}) \mid \hat{F}(z') + \hat{F}^*(z'') \geq Q\mathcal{F}(z', z'')\ \forall z', z'' \in \mathbf{R}^{nm}\}.$$

We point out that the function $Q\mathcal{F}$ is the A-quasiconvex envelope, see I. Fonseca and S. Müller [1], of the function

$$\mathcal{F}(z', z'') = \min_S \Big(F_s(z') + F_s^*(z'') \Big) \tag{2.2}$$

for the operator $A = (\mathrm{curl}, \mathrm{div})^m$.

Our main result is the following.

Theorem 2.1. *Let the hypotheses* **H1** – **H3** *hold. Then*

i) $\mathcal{A} \subset \hat{\mathcal{A}}$;

ii) *if* $\hat{F} \in \hat{\mathcal{A}}$ *then for every* $z \in \mathbf{R}^{nm}$ *there exists a sequence* $\{\sigma^k\} \subset S$ *(depending on the choice of* $\hat{F}$ *and* z*) such that*

$$F'_{\sigma^k}(z) \underset{k\to\infty}{\longrightarrow} \hat{F}'(z) \text{ as } k \to \infty.$$

3. PROOF OF THEOREM 2.1

In this Section we shall give a brief sketch of the proof of Theorem 2.1.

First of all, the smoothness and growth properties of the functions F_s and F_s^* ensure that for every given $z', z'' \in \mathbf{R}^{nm}$, $v \in H^\#$, $\eta \in N^\#$

$$\inf_{\sigma\in S} \int_K \sum_{s=1}^N \sigma_s(x)[F_s(v(x) + z') + F_s^*(\eta(x) + z'')]dx$$
$$= \int_K \min_s [F_s(v(x) + z') + F_s^*(\eta(x) + z'')]dx.$$

This ensure that the function $Q\mathcal{F}$ defined by (2.1) is the $(\mathrm{curl}, \mathrm{div})^m$-quasiconvex envelope of the function $\mathcal{F}$ defined by (2.2).

Since the functions F_s satisfy **H1** – **H3** and the functions F_s^* satisfy analogous hypotheses then almost exactly in the same way as in Miettinen and Raitums [2] it can be shown that the function $Q\mathcal{F}$ belongs to C^1. Further, from **H1** – **H3** and results by Raitums [3] it easy follows that the set $\mathcal{A}$ belongs to $E(\nu, \mu^2/\nu)$.

If the function F_σ is given as

$$F_\sigma(z') = \inf_{v\in H^\#} \int_K \sum_{s=1}^N \sigma_s(x) F_s(v(x) + z')dx$$

then

$$F^*_\sigma(z'') = \sup_{z'\in\mathbf{R}^{nm}} \big[\langle z', z''\rangle - F_\sigma(z')\big]$$
$$= \sup_{z'\in\mathbf{R}^{nm}} \sup_{v\in H^\#} \Big[-\int\limits_K \sum_{s=1}^N \sigma_s(x) F_s(v(x)+z')dx + \langle z', z''\rangle\Big]$$
$$= \inf_{\eta\in N^\#} \int\limits_K \sum_{s=1}^N \sigma_s(x) F^*_s\big(\eta(x)+z''\big)dx$$

by virtue of the representation

$$L_2(K;\mathbf{R}^{nm}) = H^\# \oplus N^\# \oplus \mathbf{R}^{nm}, \tag{3.1}$$

see, for instance, Zhikov et al. [4]. This way, for every fixed $\sigma \in S$

$$F_\sigma(z') + F^*_\sigma(z'') = \inf_{v\in H^\#} \inf_{\eta\in N^\#} \int\limits_K \sum_{s=1}^N \sigma_s(x)\Big(F_s(v(x)+z') + F^*_s(\eta(x)+z'')\Big)dx,$$

i.e. the function F_σ satisfies the inequality

$$F_\sigma(z') + F^*_\sigma(z'') \geq Q\mathcal{F}(z', z'')\ \forall z', z'' \in \mathbf{R}^{nm}. \tag{3.2}$$

That gives the inclusion $\mathcal{A} \subset \hat{\mathcal{A}}$.

We point out here, that for every pair $(z', z'') \in \mathbf{R}^{nm} \times \mathbf{R}^{nm}$ there exists a sequence $\{\sigma^k\} \subset S$ such that

$$F_{\sigma^k}(z') + F^*_{\sigma^k}(z'') \to Q\mathcal{F}(z', z'') \text{ as } k \to \infty.$$

That means that the estimate (3.2) is sharp.

By properties of conjugate functions we have that the equality

$$\hat{F}(z') = z''$$

for some $\hat{F}' \in \hat{\mathcal{A}}$ and some pair $(z', z'') \in \mathbf{R}^{nm} \times \mathbf{R}^{nm}$ is equal to the relationship

$$\hat{F}(z') + \hat{F}^*(z'') - \langle z', z''\rangle = 0. \tag{3.3}$$

The same properties of conjugate functions and the definition (2.1) of $Q\mathcal{F}$ imply

$$Q\mathcal{F}(\xi', \xi'') \geq \langle \xi', \xi''\rangle\ \forall \xi', \xi'' \in \mathbf{R}^{nm}. \tag{3.4}$$

We have used here the representation (3.1).

Since $\hat{F} \subset \hat{\mathcal{A}}$ then the relationships (3.3), (3.4) imply

$$Q\mathcal{F}(z', z'') = \langle z', z'' \rangle.$$

Because, after taking the inner infimum in (2.1)

$$Q\mathcal{F}(\xi', \xi'') = \inf_{\sigma \in S} [F_\sigma(\xi') + F_\sigma^*(\xi'')],$$

then there exists a sequence $\{\sigma^k\} \subset S$ such that

$$F_{\sigma^k}(z') + F_{\sigma^k}^*(z'') = \langle z', z'' \rangle + \varepsilon_k, \ \varepsilon_k \to +0 \text{ as } k \to \infty, \tag{3.5}$$

and, by duality,

$$F_{\sigma^k}(z') + F_{\sigma^k}^*(F'_{\sigma^k}(z')) = \langle z', F'_{\sigma^k}(z') \rangle. \tag{3.6}$$

The set $\mathcal{A}$ belongs to $E(\nu, \mu^2/\nu)$, hence, all F_σ^* belong to some class $E(\nu', \mu')$ and from (3.5) and (3.6) it follows

$$|z'' - F'_{\sigma^k}(z')| \leq \left(\frac{2\varepsilon_k}{\nu'}\right)^{1/2}, \ k \geq 1.$$

But that means $F'_{\sigma^k}(z') \to z''$ as $k \to \infty$. This way, the second statement of Theorem 2.1 is also proved.

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G-ribinių operatorių globalioji charakteristika kvazitiesinėms potencinėms elipsinėms sistemoms

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Straipsnyje nagrinėjama kvazitiesinių potencinių elipsinių sistemų šeima ir pasinaudojama, kad visi G-ribiniai operatoriai šiai šeimai gali būti charakterizuojami iškiliosios funkcijos F gradiento (lokaliai erdvinių koordinačių atžvilgiu) reikšmėmis. Parodyta, kad visos šios funkcijos F tenkina nelygybę, išreikštą per funkcijas F ir joms jungtines funkcijas.