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An H^1-Galerkin Mixed Finite Element Approximation of a Nonlocal Hyperbolic Equation

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Abstract. In this paper we investigate a semi-discrete H^1-Galerkin mixed finite element approximation of one kind of nolocal second order nonlinear hyperbolic equation, which is often used to describe vibration of an elastic string. A priori error estimates for the semi-discrete scheme are derived. A fully discrete scheme is constructed and one numerical example is given to verify the theoretical findings.

Keywords: nonlocal hyperbolic equation, H^1-Galerkin mixed finite element method, *a priori* error estimate.

AMS Subject Classification: 65N30.

1 Introduction

In this paper, we consider the following nonlocal second order nonlinear hyperbolic problem:

$$\begin{cases} u_{tt} = \Big(1 + \displaystyle\int_0^1 |u_x|^2 dx\Big) u_{xx} + f(x,t), & (x,t) \in I \times (0,T], \\ u(0,t) = 0, u(1,t) = 0, & 0 \le t \le T, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in I, \end{cases} \tag{1.1}$$

where $I = [0,1]$. $u_0(x), u_1(x)$ and $f(x,t)$ are given functions. In this model $u(x,t)$ stands for the vertical displacement of point x at instant t. This kind of equations often arise in nonlinear vibration. For more details on physical motivation we refer the interested reader to $[1,5,7,8]$ and the reference cited herein. For finite element approximation of this kind of equation one can read $[6,9,10]$.

The goal of this paper is to develop an efficient numerical method for problem (1.1). Noticing that the coefficient in (1.1) depends on the derivative of u. When finite difference method and standard Galerkin finite element method are used to solve this model, one have to calculate the derivative of the numerical solution to determine the coefficient. This would generate an inaccurate coefficient, which then reduces the accuracy of the numerical approximation for u. In order to overcome this problem we resort to mixed finite element methods, which can simultaneously approximate the unknown function $u(x,t)$ and its derivative $u_x(x,t)$. As we know the finite element spaces in classic mixed finite element methods have to satisfy the inf-sup conditions, which restricts the choosing of finite element spaces.

In this paper we utilize the H^1-Galerkin mixed finite element method to numerically solve (1.1). To our best knowledge the H^1-Galerkin mixed finite element of this kind of problem is not reported in the literatures. The H^1-Galerkin mixed finite element method was proposed in [11] for parabolic problems, which can be viewed as a non-symmetric version of least square method. Compared with standard mixed finite element method H^1-Galerkin mixed finite element method does not require the finite element spaces to satisfy the inf-sup conditions, which makes the choosing of finite element spaces more flexible. It has been proved that the H^1-Galerkin mixed finite element method has the same rate of convergence as standard mixed finite element method. For more references with respect to H^1-Galerkin mixed finite element method one can refer to [2, 3, 4, 11, 12, 13, 15].

By introducing a new variable $q = u_x$ we split problem (1.1) into a first order system. Then we construct a semi-discrete H^1-Galerkin mixed finite element approximate scheme. A priori error estimates for unknown function u in L^2 and H^1 norm, and q in L^2 norm are derived respectively. In order to carry out numerical experiment we also construct a fully discrete scheme, where the backward Euler method is used to discretize the time variable. Finally a numerical example is given to verify the theoretical findings.

The rest of this paper is organized as follows: In Section 2 a semi-discrete H^1-Galerkin mixed finite element approximate scheme is constructed. Optimal a priori error estimates are deduced in Section 3. In Section 4 a fully discrete scheme based on the backward Euler method is developed and a numerical example is presented to illustrate our theoretical results.

Throughout the paper, we use the standard notation $W^{m,q}(I)$ for Sobolev space on I with a norm $\|\cdot\|_{m,q}$ and a semi-norm $|\cdot|_{m,q}$. For $q=2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $\|\cdot\|_m = \|\cdot\|_{m,2}$ and for $m=0$, we denote $\|\cdot\| = \|\cdot\|_0$. The inner products in $L^2(I)$ are indicated by $(\cdot,\cdot)$.

For the spaces involving time we introduce the following notations. Let X be a Banach space and $\varphi(t) : [0,T] \longmapsto X$, we set

$$\|\varphi\|_{L^2(0,T;X)} = \Big(\int_0^T \|\varphi(s)\|_X^2 ds\Big)^{\frac{1}{2}}, \quad \|\varphi\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{\mathbf{0\le t\le T}} \|\varphi\|_{\mathbf{X}}.$$

In addition, C denotes a generic constant independent of the spatial mesh parameter h.

2 Semi-discrete H^1-Galerkin mixed finite element scheme

In this section we first derive the variational formulation for problem (1.1), and then construct a semi-discrete H^1-Galerkin mixed finite element scheme.

In order to define a H^1-Galerkin mixed finite element procedure for problem (1.1), we decompose (1.1) into a first order system. Let $q = u_x$, then (1.1) reduces to

$$\begin{cases} q = u_x, \\ u_{tt} = (1 + \omega(q))q_x + f. \end{cases} \tag{2.1}$$

Here $\omega(q) := \int_0^1 q^2 dx$. Let $H_0^1(I) = \{v \in H^1(I); v(0) = v(1) = 0\}$. Multiplying the equation (2.1) by v_x with $v \in H_0^1(I)$, and integrating on interval I gives

$$(u_x, v_x) = (q, v_x),\ v \in H_0^1(I). \tag{2.2}$$

In a similar way we deduce

$$(u_{tt}, w_x) = ((1 + \omega(q))q_x, w_x) + (f, w_x), \quad w \in H^1(I). \tag{2.3}$$

By Green formula and $u_{tt}(0, t) = u_{tt}(1, t) = 0$, we obtain

$$(u_{tt}, w_x) = -(q_{tt}, w).$$

Then (2.3) becomes

$$(q_{tt}, w) + ((1 + \omega(q))q_x, w_x) + (f, w_x) = 0, \quad w \in H^1(I). \tag{2.4}$$

Collecting (2.2) and (2.4), we arrive at the weak formulation for problem (1.1): find $(u, q) \in H_0^1(I) \times H^1(I)$ satisfying

$$\begin{cases} (u_x, v_x) = (q, v_x), \quad v \in H_0^1(I), \\ (q_{tt}, w) + ((1 + \omega(q))q_x, w_x) + (f, w_x) = 0, \quad w \in H^1(I). \end{cases} \tag{2.5}$$

Let $V_h \subset H_0^1(I), W_h \subset H^1(I)$ be the finite element spaces consisting of piecewise polynomials of order k and r, respectively, and satisfying the following approximation properties:

$$\inf_{\psi_h \in V_h} \{\|\psi - \psi_h\|_{0,p} + h\|\psi - \psi_h\|_{1,p}\} \le Ch^{k+1}\|\psi\|_{k+1,p},\ \psi \in H_0^1(I) \cap W^{k+1,p}(I),$$
$$\inf_{w_h \in W_h} \{\|w - w_h\|_{0,p} + h\|w - w_h\|_{1,p}\} \le Ch^{r+1}\|w\|_{r+1,p}, \quad w \in W^{r+1,p}(I),$$

where $1 \le p \le \infty$. k, r are positive integers.

Then the semi-discrete H^1-Galerkin mixed finite element approximation of (1.1) can be characterized as finding $(u_h, q_h) \in V_h \times W_h$ such that

$$\begin{cases} (u_{hx}, v_{hx}) = (q_h, v_{hx}), \quad v_h \in V_h, \\ (q_{htt}, w_h) + ((1 + \omega(q_h))q_{hx}, w_{hx}) + (f, w_{hx}) = 0, \quad w_h \in W_h, \end{cases} \tag{2.6}$$

with given initial value $q_h(0), q_{ht}(0)$ and $\omega(q_h) = \int_0^1 q_h^2 dx$.

3 Error Analysis

To derive the error estimate we decompose the errors as

$$u - u_h = u - \widetilde{u}_h + \widetilde{u}_h - u_h = \rho_u + \xi_u,$$
$$q - q_h = q - \widetilde{q}_h + \widetilde{q}_h - q_h = \rho_q + \xi_q,$$

where $\widetilde{u}_h \in V_h$ and $\widetilde{q}_h \in W_h$ are elliptic projections defined by

$$(u_x - \widetilde{u}_{hx}, v_{hx}) = 0, \qquad \forall v_h \in V_h, \tag{3.1}$$
$$(q_x - \widetilde{q}_{hx}, w_{hx}) + \alpha(q - \widetilde{q}_h, w_h) = 0, \qquad \forall w_h \in W_h. \tag{3.2}$$

Here α is chosen to guarantee the H^1-coercivity of the bilinear form in the second equation. Moreover, it is easy to check that the bilinear form is bounded. According to [14], we have the following estimates with $j = 0, 1$ and $t \in [0, T]$

$$\| \rho_u \|_j + \| \frac{\partial \rho_u}{\partial t} \|_j \leq C h^{k+1-j} (\| u \|_{k+1} + \| u_t \|_{k+1}),$$
$$\| \rho_q \|_j + \| \frac{\partial \rho_q}{\partial t} \|_j + \| \frac{\partial^2 \rho_q}{\partial t^2} \|_j \leq C h^{r+1-j} (\| q \|_{r+1} + \| q_t \|_{r+1} + \| q_{tt} \|_{r+1}).$$

Using (2.5), (2.6) and (3.1)–(3.2), we can deduce the following error equations:

$$(\frac{\partial \xi_u}{\partial x}, v_{hx}) = (\rho_q, v_{hx}) + (\xi_q, v_{hx}), \tag{3.3}$$
$$\begin{aligned}(\frac{\partial^2 \xi_q}{\partial t^2}, w_h) + (\frac{\partial \xi_q}{\partial x}, w_{hx}) &= -(\frac{\partial^2 \rho_q}{\partial t^2}, w_h) + (\alpha \rho_q, w_h) \\ &\quad - (\omega(q) q_x - \omega(q_h) q_{hx}, w_{hx}).\end{aligned} \tag{3.4}$$

Theorem 1. *Suppose that $q_h(0) = \widetilde{q}_h(0), q_{ht}(0) = \widetilde{q}_{ht}(0)$. Then there exists a positive constant C independent of h such that*

$$\| u - u_h \| + h \| u - u_h \|_1 + \| q - q_h \| \leq C h^{\min\{k+1,\ r+1\}}.$$

Proof. Choosing $v_h = \xi_u$ in (3.3) yields

$$(\frac{\partial \xi_u}{\partial x}, \frac{\partial \xi_u}{\partial x}) = (\rho_q, \frac{\partial \xi_u}{\partial x}) + (\xi_q, \frac{\partial \xi_u}{\partial x}),$$

which implies

$$\|\frac{\partial \xi_u}{\partial x}\| \leq \|\rho_q\| + \|\xi_q\|. \tag{3.5}$$

Setting $w_h = \frac{\partial \xi_q}{\partial t}$ in (3.4) gives

$$\begin{aligned}(\frac{\partial^2 \xi_q}{\partial t^2}, \frac{\partial \xi_q}{\partial t}) + (\frac{\partial \xi_q}{\partial x}, \frac{\partial^2 \xi_q}{\partial x \partial t}) &= -(\frac{\partial^2 \rho_q}{\partial t^2}, \frac{\partial \xi_q}{\partial t}) + (\alpha \rho_q, \frac{\partial \xi_q}{\partial t}) \\ &\quad - (\omega(q) q_x - \omega(q_h) q_{hx}, \frac{\partial^2 \xi_q}{\partial x \partial t}).\end{aligned}$$

Then we conclude

$$\frac{1}{2}\frac{d}{dt}(\|\frac{\partial \xi_q}{\partial t}\|^2+\|\frac{\partial \xi_q}{\partial x}\|^2)=-(\frac{\partial^2\rho_q}{\partial t^2},\frac{\partial \xi_q}{\partial t})+(\alpha\rho_q,\frac{\partial \xi_q}{\partial t})$$
$$-(\omega(q)q_x-\omega(q_h)q_{hx},\frac{\partial^2\xi_q}{\partial x\partial t}).$$

Note that

$$\begin{aligned}\omega(q)q_x-\omega(q_h)q_{hx}&=\omega(q)(q_x-\widetilde{q}_{hx})+(\omega(q)-\omega(q_h))\widetilde{q}_{hx}+\omega(q_h)(\widetilde{q}_{hx}-q_{hx})\\&=\omega(q)\frac{\partial\rho_q}{\partial x}+(\omega(q)-\omega(q_h))\widetilde{q}_{hx}+\omega(q_h)\frac{\partial\xi_q}{\partial x}.\end{aligned}$$

Then we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\frac{\partial \xi_q}{\partial t}\|^2+\|\frac{\partial \xi_q}{\partial x}\|^2)&=-(\frac{\partial^2\rho_q}{\partial t^2},\frac{\partial \xi_q}{\partial t})+(\alpha\rho_q,\frac{\partial \xi_q}{\partial t})-(\omega(q)\frac{\partial\rho_q}{\partial x},\frac{\partial^2\xi_q}{\partial x\partial t})\\&-((\omega(q)-\omega(q_h))\widetilde{q}_{hx},\frac{\partial^2\xi_q}{\partial x\partial t})-(\omega(q_h)\frac{\partial\xi_q}{\partial x},\frac{\partial^2\xi_q}{\partial x\partial t}).\end{aligned}\tag{3.6}$$

By the definition of elliptic projection $\widetilde{q}_h$ we derive

$$\begin{aligned}&(\omega(q)\frac{\partial\rho_q}{\partial x},\frac{\partial^2\xi_q}{\partial x\partial t})=-(\alpha\omega(q)\rho_q,\frac{\partial\xi_q}{\partial t}),\\&((\omega(q)-\omega(q_h))\widetilde{q}_{hx},\frac{\partial^2\xi_q}{\partial x\partial t})=\frac{d}{dt}((\omega(q)-\omega(q_h))\widetilde{q}_{hx},\frac{\partial\xi_q}{\partial x})\\&\qquad-(\frac{d}{dt}[(\omega(q)-\omega(q_h))\widetilde{q}_{hx}],\frac{\partial\xi_q}{\partial x})\\&(\omega(q_h)\frac{\partial\xi_q}{\partial x},\frac{\partial^2\xi_q}{\partial x\partial t})=\frac{1}{2}\frac{d}{dt}(\omega(q_h)\frac{\partial\xi_q}{\partial x},\frac{\partial\xi_q}{\partial x})-\frac{1}{2}(\frac{d}{dt}\omega(q_h)\frac{\partial\xi_q}{\partial x},\frac{\partial\xi_q}{\partial x}).\end{aligned}$$

Inserting above terms into (3.6) leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\frac{\partial \xi_q}{\partial t}\|^2+\|\frac{\partial \xi_q}{\partial x}\|^2+(\omega(q_h)\frac{\partial\xi_q}{\partial x},\frac{\partial\xi_q}{\partial x}))&=-(\frac{\partial^2\rho_q}{\partial t^2},\frac{\partial \xi_q}{\partial t})\\&+(\alpha(1+\omega(q))\rho_q,\frac{\partial \xi_q}{\partial t})-\frac{d}{dt}((\omega(q)-\omega(q_h))\widetilde{q}_{hx},\frac{\partial\xi_q}{\partial x})\\&+(\frac{d}{dt}[(\omega(q)-\omega(q_h))\widetilde{q}_{hx}],\frac{\partial\xi_q}{\partial x})+\frac{1}{2}(\frac{d}{dt}\omega(q_h)\frac{\partial\xi_q}{\partial x},\frac{\partial\xi_q}{\partial x}).\end{aligned}$$

Integrating above equation from 0 to t and using $\xi_q(0)=0,\frac{\partial\xi_q}{\partial t}(0)=0$ we arrive at

$$\begin{aligned}\frac{1}{2}\Big(\|\frac{\partial \xi_q}{\partial t}\|^2+\|\frac{\partial \xi_q}{\partial x}\|^2+(\omega(q_h)\frac{\partial\xi_q}{\partial x},\frac{\partial\xi_q}{\partial x})\Big)&=-\int_0^t(\frac{\partial^2\rho_q}{\partial t^2},\frac{\partial \xi_q}{\partial t})ds\\&+\int_0^t(\alpha(1+\omega(q))\rho_q,\frac{\partial \xi_q}{\partial t})ds-((\omega(q)-\omega(q_h))\widetilde{q}_{hx},\frac{\partial\xi_q}{\partial x})\\&+\frac{1}{2}\int_0^t(\frac{d}{dt}\omega(q_h)\frac{\partial\xi_q}{\partial x},\frac{\partial\xi_q}{\partial x})ds+\int_0^t(\frac{d}{dt}[(\omega(q)-\omega(q_h))\widetilde{q}_{hx}],\frac{\partial\xi_q}{\partial x})ds.\end{aligned}$$

In the following we will derive the estimates of the terms on the right hand side. By Cauchy-Schwarz inequality we deduce

$$\int_0^t (\frac{\partial^2 \rho_q}{\partial t^2}, \frac{\partial \xi_q}{\partial t}) ds \le \frac{1}{2}\int_0^t \|\frac{\partial^2 \rho_q}{\partial t^2}\|^2 ds + \frac{1}{2}\int_0^t \|\frac{\partial \xi_q}{\partial t}\|^2 ds \tag{3.7}$$

$$\int_0^t (\alpha(1+\omega(q))\rho_q, \frac{\partial \xi_q}{\partial t}) ds \le C\int_0^t \|\rho_q\|^2 ds + C\int_0^t \|\frac{\partial \xi_q}{\partial t}\|^2 ds. \tag{3.8}$$

Using ϵ-inequality and Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&|((\omega(q)-\omega(q_h))\widetilde{q}_{hx}, \frac{\partial \xi_q}{\partial x})| \le |\omega(q)-\omega(q_h)|\|\widetilde{q}_{hx}\|\|\frac{\partial \xi_q}{\partial x}\| \\
&\quad \le C\|\frac{\partial \xi_q}{\partial x}\|\|q-q_h\|\|q+q_h\| \\
&\quad \le C\|\frac{\partial \xi_q}{\partial x}\|(\|\rho_q\|+\|\xi_q\|)(\|q\|+\|\widetilde{q}_h\|+\|\xi_q(t)\|) \\
&\quad \le \varepsilon\|\frac{\partial \xi_q}{\partial x}\|^2 + C(\|\xi_q(t)\|_{L^\infty(0,t;L^2(I))})(\|\rho_q\|^2 + \int_0^t \|\frac{\partial \xi_q}{\partial t}\|^2 ds).
\end{aligned} \tag{3.9}$$

Here the boundness of $\|\widetilde{q}_h\|$ and $\|\widetilde{q}_{hx}\|$ as well as inequality $\|\xi_q\| \le \int_0^t \|\frac{\partial \xi_q}{\partial t}\| ds$ were used. $C(\|\xi_q(t)\|_{L^\infty(0,t;L^2(I))})$ is a constant depending on $\|\xi_q(t)\|$. Utilizing Hölder inequality we obtain

$$\begin{aligned}
&\int_0^t (\frac{d}{dt}\omega(q_h)\frac{\partial \xi_q}{\partial x}, \frac{\partial \xi_q}{\partial x}) ds = 2\int_0^t \int_I q_h q_{ht} dx \|\frac{\partial \xi_q}{\partial x}\|^2 ds \\
&\quad \le 2\int_0^t \|q_h\|\,\|q_{ht}\|\,\|\frac{\partial \xi_q}{\partial x}\|^2 ds \\
&\quad \le C(\|\xi_q\|_{L^\infty(0,t;L^2(I))}, \|\frac{\partial \xi_q}{\partial t}\|_{L^\infty(0,t;L^2(I))}) \int_0^t \|\frac{\partial \xi_q}{\partial x}\|^2 ds.
\end{aligned} \tag{3.10}$$

For the last term we have

$$\begin{aligned}
\int_0^t (\frac{d}{dt}[(\omega(q)-\omega(q_h))\widetilde{q}_{hx}], \frac{\partial \xi_q}{\partial x}) ds &= \int_0^t ((\omega(q)-\omega(q_h))\frac{\partial \widetilde{q}_{hx}}{\partial t}, \frac{\partial \xi_q}{\partial x}) ds \\
&\quad + \int_0^t (\frac{d}{dt}[\omega(q)-\omega(q_h)]\widetilde{q}_{hx}, \frac{\partial \xi_q}{\partial x}) ds.
\end{aligned}$$

Using the boundness of $\|\widetilde{q}_h\|$ and $\|\widetilde{q}_{hxt}\|$ we derive

$$\begin{aligned}
&\left|\int_0^t ((\omega(q)-\omega(q_h))\frac{\partial \widetilde{q}_{hx}}{\partial t}, \frac{\partial \xi_q}{\partial x}) ds\right| \le \int_0^t \|\widetilde{q}_{hxt}\||\omega(q)-\omega(q_h)|\|\frac{\partial \xi_q}{\partial x}\| ds \\
&\quad \le C\int_0^t \|\frac{\partial \xi_q}{\partial x}\|\|q-q_h\|\|q+q_h\| ds \\
&\quad \le C\int_0^t \|\frac{\partial \xi_q}{\partial x}\|(\|\rho_q\|+\|\xi_q\|)(\|q\|+\|\widetilde{q}_h\|+\|\xi_q(t)\|) ds \\
&\quad \le C(\|\xi_q\|_{L^\infty(0,t;L^2(I))}) \int_0^t (\|\frac{\partial \xi_q}{\partial x}\|^2 + \|\rho_q\|^2 + \|\frac{\partial \xi_q}{\partial t}\|^2) ds.
\end{aligned}$$

By Hölder inequality we deduce

$$|\frac{d}{dt}[(\omega(q)-\omega(q_h))]| \le C(\|\rho_q\|+\|\xi_q\|) + C(\|\xi_q\|_{L^\infty(0,t;L^2(I))})(\|\frac{\partial\rho_q}{\partial t}\| + \|\frac{\partial\xi_q}{\partial t}\|).$$

Then we have

$$\int_0^t(\frac{d}{dt}[(\omega(q)-\omega(q_h))]\widetilde{q}_{hx}, \frac{\partial\xi_q}{\partial x})ds \le C(\|\xi_q\|_{L^\infty(0,t;L^2(I))})\int_0^t(\|\frac{\partial\rho_q}{\partial t}\|^2+\|\frac{\partial\xi_q}{\partial t}\|^2 + \|\frac{\partial\xi_q}{\partial x}\|^2)ds + C\int_0^t(\|\frac{\partial\xi_q}{\partial x}\|^2 + \|\frac{\partial\xi_q}{\partial t}\|^2 + \|\rho_q\|^2)ds. \tag{3.11}$$

Using (3.7)–(3.11) we derive

$$\begin{aligned}\frac{1}{2}\Big(\|\frac{\partial\xi_q}{\partial t}\|^2 + \|\frac{\partial\xi_q}{\partial x}\|^2\Big) &\le C(\|\xi_q\|_{L^\infty(0,t;L^2(I))})(\|\rho_q\|^2 \\ &+ \int_0^t(\|\frac{\partial\xi_q}{\partial t}\|^2 + \|\frac{\partial\rho_q}{\partial t}\|^2 + \|\frac{\partial\xi_q}{\partial x}\|^2)ds) \\ &+ C(\|\xi_q\|_{L^\infty(0,t;L^2(I))}, \|\frac{\partial\xi_q}{\partial t}\|_{L^\infty(0,t;L^2(I))})\int_0^t\|\frac{\partial\xi_q}{\partial x}\|^2 ds \\ &+ C\int_0^t(\|\rho_q\|^2 + \|\frac{\partial^2\rho_q}{\partial t^2}\|^2 + \|\frac{\partial\xi_q}{\partial t}\|^2 + \|\frac{\partial\xi_q}{\partial x}\|^2)ds + \epsilon\|\frac{\partial\xi_q}{\partial x}\|^2.\end{aligned}$$

In order to prove the theorem result, we need to make the following induction hypothesis: there exists a constant $0 < h_0 < 1$ such that the following estimate holds for $0 < h \le h_0$:

$$\max\{\|\xi_q\|_{L^\infty(0,t;L^2(I))}, \|\frac{\partial\xi_q}{\partial t}\|_{L^\infty(0,t;L^2(I))}\} < 1, 0 \le t \le T.$$

Then using above inequality and setting ε small enough we derive

$$\begin{aligned}\|\frac{\partial\xi_q}{\partial t}\|^2 + \|\frac{\partial\xi_q}{\partial x}\|^2 \quad &\le \quad C\int_0^t(\|\rho_q\|^2 + \|\frac{\partial\rho_q}{\partial t}\|^2 + \|\frac{\partial^2\rho_q}{\partial t^2}\|^2)ds + C\|\rho_q\|^2 \\ &+C\int_0^t(\|\frac{\partial\xi_q}{\partial x}\|^2 + \|\frac{\partial\xi_q}{\partial t}\|^2)ds.\end{aligned}$$

By Gronwall's Lemma we obtain

$$\|\frac{\partial\xi_q}{\partial t}\|^2 + \|\frac{\partial\xi_q}{\partial x}\|^2 \le C\int_0^t(\|\rho_q\|^2 + \|\frac{\partial\rho_q}{\partial t}\|^2 + \|\frac{\partial^2\rho_q}{\partial t^2}\|^2)ds + C\|\rho_q\|^2.$$

Using the estimates of elliptic projection we further deduce

$$\|\frac{\partial\xi_q}{\partial t}\| + \|\frac{\partial\xi_q}{\partial x}\| \quad \le \quad \mathcal{C}h^{r+1}. \tag{3.12}$$

Here $\mathcal{C} > 0$ is independent of h.

Now we are in position to prove the induction hypothesis. We suppose that there exists a constant $0 < h_* \le h_0$ such that

$$\max\{\|\xi_q^*\|_{L^\infty(0,t;L^2(I))}, \|\frac{\partial\xi_q^*}{\partial t}\|_{L^\infty(0,t;L^2(I))}\} \ge 1, \quad 0 \le t \le T.$$

We define t^* by

$$t^* = \inf\{t \in [0,T] | \max\{\|\xi_q^*\|_{L^\infty(0,t;L^2(I))},\ \|\frac{\partial \xi_q^*}{\partial t}\|_{L^\infty(0,t;L^2(I))}\} \geq 1\}.$$

Then we have

$$\begin{aligned} &\max\{\|\xi_q^*\|_{L^\infty(0,t^*;L^2(I))},\ \ \|\frac{\partial \xi_q^*}{\partial t}\|_{L^\infty(0,t^*;L^2(I))}\} = 1, \\ &\max\{\|\xi_q^*\|_{L^\infty(0,t;L^2(I))},\ \ \|\frac{\partial \xi_q^*}{\partial t}\|_{L^\infty(0,t;L^2(I))} \leq 1\}, \quad 0 < t \leq t^*. \end{aligned} \tag{3.13}$$

By the argument similar to (3.12) we can prove

$$\|\frac{\partial \xi_q^*}{\partial t}\| + \|\frac{\partial \xi_q^*}{\partial x}\| \quad \leq \quad \mathcal{C}_1 h^{r+1}, \quad 0 < t \leq t^*.$$

Furthermore, using above estimate we can derive

$$\|\xi_q^*\| \leq C \int_0^t \|\frac{\partial \xi_q^*}{\partial t}\| ds \leq \mathcal{C}_2 h^{r+1}.$$

Choose h_0 such that

$$\max\{\mathcal{C}_1 h_0^{r+1}, \mathcal{C}_2 h_0^{r+1}\} \leq 1/2.$$

Then we have

$$\max\{\|\xi_q^*\|_{L^\infty(0,t^*;L^2(I))}, \|\frac{\partial \xi_q^*}{\partial t}\|_{L^\infty(0,t^*;L^2(I))}\} \leq \frac{1}{2}.$$

This contradicts with (3.13). Thus the induction hypothesis holds.

Substituting (3.12) into (3.5) leads to

$$\|\frac{\partial \xi_u}{\partial x}\|^2 \leq C\|\rho_q\|^2 + C\int_0^t \|\frac{\partial \xi_q}{\partial t}\|^2 ds \leq C h^{r+1}. \tag{3.14}$$

Note that

$$\|\xi_q\| \leq C \int_0^t \|\frac{\partial \xi_q}{\partial t}\| ds, \quad \|\xi_u\| \leq \|\xi_u\|_1 \leq C\|\frac{\partial \xi_u}{\partial x}\|.$$

Then we can derive the theorem result by combining (3.12), (3.14) and the estimates of ρ_u, ρ_q, and using the triangle inequality. □

4 Numerical example

The goal of this section is to carry out a numerical example to illustrate the theoretical findings presented in Section 3.

Let $T = 1$ and the exact solution u is chosen as

$$u(x,t) = \sin(\pi x)\sin^3(t), \quad (x,t) \in [0,1] \times [0,1].$$

The corresponding right hand term f and initial values $u_0(x), u_1(x)$ can be calculated by the governing equations.

Let $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ be a time grid with $\tau = t_n - t_{n-1},\ n = 1, 2, \cdots, N$. We set:

$$\bar{\partial}_t^2 \psi^n = \frac{\psi^{n+1} - 2\psi^n + \psi^{n-1}}{\tau^2}, \quad \psi^{n,\frac{1}{4}} = \frac{\psi^{n+1} + 2\psi^n + \psi^{n-1}}{4}.$$

For the discretization of time derivative we adopt the second order backward Euler scheme. Then for given initial value Q^0, Q^1, the fully discrete H^1-Galerkin mixed finite element approximation of (1.1) is defined by

$$\begin{cases} (U_x^n, v_{hx}) = (Q^n, v_{hx}), \quad v_h \in V_h, \\ (\bar{\partial}_t^2 Q^n, w_h) + ((1+\omega(Q^n))Q_x^{n,\frac{1}{4}}, w_{hx}) + (f^{n,\frac{1}{4}}, w_{hx}) = 0, \quad w_h \in W_h. \end{cases}$$

In the numerical experiment the unknown function $u(x,t)$ and its derivative $q(x,t)$ are approximated by piecewise linear polynomials, i.e., $k = r = 1$.

The errors for $u - U$ and $q - Q$ at different time are displayed in Tables 1, 2, and 3. The order of convergence at $t = 0.4, 0.6$ are shown in Figure 1. We can observe that the orders of convergence are in agreement with our theoretical findings presented in Section 3. The surface of U and Q are presented in Figure 2.

Table 1. The errors of $\| u^n - U_h^n \|$ at different time.

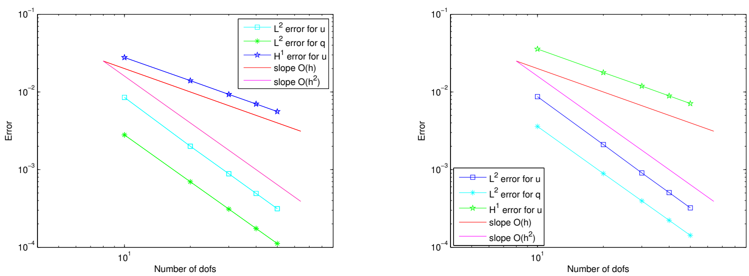
$h = \Delta t$	Time $t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1.0$
1/10	0.0014	0.0028	0.0036	7.0089e-004	0.0088
1/20	3.5322e-004	6.9799e-004	8.9011e-004	1.6026e-004	0.0022
1/30	1.5709e-004	3.1037e-004	3.9546e-004	7.0033e-005	9.7655e-004
1/40	1.5709e-004	3.1037e-004	3.9546e-004	7.0033e-005	9.7655e-004
1/50	5.6573e-005	1.1176e-004	1.4234e-004	2.4993e-005	3.5139e-004

Table 2. The errors of $\| u^n - U_h^n \|_1$ at different time.

$h = \Delta t$	Time $t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1.0$
1/10	0.0141	0.0278	0.0357	0.0070	0.0885
1/20	0.0071	0.0140	0.0178	0.0032	0.0440
1/30	0.0047	0.0093	0.0119	0.0021	0.0293
1/40	0.0035	0.0070	0.0089	0.0016	0.0220
1/50	0.0028	0.0056	0.0071	0.0012	0.0176

Table 3. The errors of $\parallel q^n - Q_h^n \parallel$ at different time.

$h = \Delta t$	Time $t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1.0$
1/10	0.0047	0.0085	0.0087	0.0050	0.0427
1/20	0.0011	0.002	0.0021	0.0012	0.0101
1/30	4.9371e-004	8.8399e-004	9.0643e-004	5.4739e-004	0.0044
1/40	2.7549e-004	4.9325e-004	5.0559e-004	3.0609e-004	0.0025
1/50	1.7546e-004	3.1414e-004	3.2195e-004	1.9514e-004	0.0016



a) b)

Figure 1. The convergence rates for u and q: a) at $t = 0.4$, b) $t = 0.6$

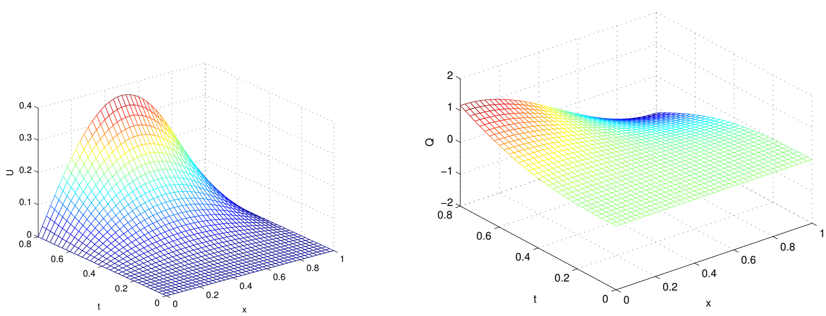


Figure 2. The surface for U and Q on $[0, 1] \times [0, 0.8]$

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