Uniformly Valid Asymptotics for Carrier's Mathematical Model of String Oscillations

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Abstract. In the paper, an asymptotic analysis of G.F. Carrier's mathematical model of string oscillation is presented. The model consists of a system of two nonlinear second order partial differential equations and periodic initial conditions. The longitudinal and transversal string oscillations are analyzed together when at the initial moment of time the system's solutions have amplitudes proportional to a small parameter. The problem is reduced to a system of two weakly nonlinear wave equations. The resonant interaction of periodic waves is analyzed. An uniformly valid asymptotic approximation in the long time interval, which is inversely proportional to the small parameter, is constructed. This asymptotic approximation is a solution of averaged along characteristics integro-differential system. Conditions of appearance of combinatoric resonances in the system have been established. The results of numerical experiments are presented.

Keywords: asymptotic analysis, averaging, nonlinear waves, resonances.

AMS Subject Classification: 35L60; 41A60; 74J30.

1 The mathematical model

The linear string oscillation equation is

$$v_{tt} - b^2 v_{xx} = 0, \qquad (1.1)$$

here $x \in [0, l]$ is a coordinate, $t \in [0, \infty)$ – time, $b > 0$ – sound velocity, and $v(x, t)$ is the string transverse deviation from the equilibrium position which, together with the initial and boundary conditions, makes the well-known small-amplitude string fluctuation model (see, e. g., [30]).

Equation (1.1) is obtained by introducing certain mathematical and physical preconditions limiting the applicability of the model. For example, mathematical preconditions regarding the smallness of the function $v(x,t)$: $|v(x,t)| \ll 1$, analyticity or at least smoothing $v(x,t) \in C^2([0,l] \times [0,\infty))$, physical preconditions regarding string weightlessness, nontensility and absolute tensility. In the opposite case, we should account for the effect of gravitational forces and for the fact that the string's length in the process of its motions may change, therefore the Hooke tensility law does not work.

If we reject at least one above-mentioned or other "obvious" precondition, we shall obtain various generalizations of the linear wave equation (1.1) and of the related model.

For example, upon rejecting the conditions of the small deviation and given that $v(x,t) \sim 1$, the equation of motion begins to depend on the first-order derivative $v_x(x,t)$ and becomes nonlinear. We can show that in this case from the Lagrangian function, corresponding to such an oscillating string

$$\mathcal{L} = \frac{1}{2}\rho A v_t^2 - T(\sqrt{1+v_x^2} - 1)$$

with the usual boundary conditions $v(0,t) = v(l,t) = 0$, there follows a more complicated equation of motion

$$v_{tt} = \frac{b^2 v_{xx}}{(1+v_x^2)^{3/2}}, \quad b = \frac{T}{\rho A}. \tag{1.2}$$

Employing mechanical reasonings, this equation was obtained [22].

A broader context of modelling is presented in the monographs [12, 31]. The monograph [31] deals with the general properties of the mathematical modelling whereas the monograph [12] helps us to find asymptotic solutions of the strongly nonlinear systems of differential equations. The monograph is dedicated to a very important fundamental problem: the inversion of Lagranges theorem on the stability of equilibrium not a trivial task, over which researchers had struggled for more than half a century.

The oscillation equations of the both mentioned models have their regions of validity. Equation (1.2) is nonlinear, valid also in the case when the value of the transversal deviations $|v(x,t)| \gg 1$. However, in the case transversal deviations are too large, equation (1.2) becomes misleading: in case of large deviations, the deflection of a string element may be not only transversal, but also longitudinal. This means that large deviations are characterized by two functions: transversal $v(x,t)$ and longitudinal $u(x,t)$. It is the idea that has stimulated a number of authors to formulate the corresponding nonlinear models of a fluctuating string not for one but for two functions – $v(x,t)$ and $u(x,t)$ (see, e. g., [1, 2, 7, 21]).

For example, this attracted G.F.Carrier's attention in [5,6] (see also [8,25]); he has formulated a model of the nonlinear motion of a string, in which he analyzes both transversal and longitudinal string oscillations, describing them

by a system of two nonlinear second-order partial differential equations:

$$\begin{cases} \rho A u_{tt} - EA u_{xx} - (EA - T)\dfrac{(1 + u_x)v_x v_{xx} - u_{xx}v_x^2}{\sqrt{[(1 + u_x)^2 + v_x^2]^3}} = 0\,, \\[4mm] \rho A v_{tt} - EA v_{xx} + (EA - T)\dfrac{(1 + u_x)^2 v_{xx} - (1 + u_x)u_{xx}v_x}{\sqrt{[(1 + u_x)^2 + v_x^2]^3}} = 0\,, \end{cases} \tag{1.3}$$

here ρ is the substance density, A – the transection area, E – the Young modulus of string, T – the initial tension. Differently from the classical rectilinear motion, an element of a non-stimulated string is characterized by the spatial axial coordinate x and the time moment t. Deviations of the stimulated string element are characterized by the longitudinal $u(x,t)$ and transverse $v(x,t)$ displacements. The string moves in the (u, v) plane. Such a note may be regarded as a parametrical curve which depends not only on the spatial but also on the time coordinate for which the radius-vector $r = (u(x,t), v(x,t))$.

What is the interrelation of the equations of motion (1.1), (1.2) and (1.3)? In the absence of longitudinal oscillations, i.e. when $u(x,t) \equiv u_x(x,t) \equiv u_{xx}(x,t) \equiv 0$, system (1.3) degenerates into one equation which characterizes only transverse oscillations $v(x,t)$:

$$\rho A v_{tt} - \frac{T v_{xx}}{(1 + v_x^2)^{3/2}} - EA\left[v_{xx} - \frac{v_{xx}}{(1 + v_x^2)^{3/2}}\right] = 0\,. \tag{1.4}$$

In case when the condition

$$v_x^2 \ll \frac{2}{3}\frac{T}{EA} \tag{1.5}$$

is valid, the last term in equation (1.4) may be ignored, and this equation transforms into equation (1.2). Indeed, when $|v_x^2| \ll 1$, using the series

$$(1 + v_x^2)^{3/2} = 1 + \frac{3}{2}v_x^2 + O(v_x^4)\,, \tag{1.6}$$

equation (1.4) may be written as follows:

$$\rho A v_{tt} - \frac{T v_{xx}}{(1 + v_x^2)^{3/2}} - \frac{\frac{3}{2}EA v_{xx} v_x^2}{(1 + v_x^2)^{3/2}} = 0\,.$$

The second term will be much larger than the third one if its numerator will be significantly larger than the third term numerator, i.e.

$$T \gg \frac{3}{2}EA v_x^2\,, \tag{1.7}$$

and it is from here that the condition (1.5) follows.

It is well known (see, e. g., [4]) that $\dfrac{T}{EA}$ is a deformation or, in our case, the relative lengthening of the string $\dfrac{\Delta l}{l}$. Condition (1.5) has a simple geometrical interpretation:

$$v_x^2 \ll \frac{2}{3}\frac{\Delta l}{l}\,. \tag{1.8}$$

Inequality (1.8) and expansion (1.6) will be satisfied if $\Delta l < \frac{3}{2}\, l$.

Let us summarize: when a swaying string lengthens more than $3/2$ times, the Carrier's model (1.3) should be used. When the lengthening does not exceed more than $3/2$ of the string length but is rather large, e.g., $\Delta l \sim l$, model (1.2) may be applied. Last but not least, when the condition $\Delta l \ll l$ and correspondingly $v_x^2 \ll 1$, equation (1.2) turns into the classical rectilinear swaying equation (1.1). Note that the presented limitations of the application of the models are valid in the bounded region of changing the independent variables $(x,t) \in [0,l] \times \left[0,t^0\right]$, when the constants l, t^0 do not depend on the asymptotic relations (1.5), (1.7), etc. The more subtle asymptotic analysis performed in this paper shows that when a large (x,t) region of variable changes is analyzed, transition to a simpler model may be groundless even at small oscillation amplitudes.

Coming back to system (1.3), let us note that when at the initial moment of time $t = 0$ the system solutions have small amplitudes (proportional to the small parameter ε), the problem is reduced to a system of two nonlinear wave equations. In this paper, we construct an asymptotic approximation uniformly valid in the long time interval $t \sim \varepsilon^{-1}$ and determine the appearance conditions of the resonant wave interactions.

2   Analysis of the model by the small parameter method

Suppose that at the initial moment of time $t = 0$ the following conditions are satisfied:

$$u(x,0) = \tilde{\varepsilon} u_0\left(x\right),\ u_t\left(x,0\right) = \tilde{\varepsilon} u_1\left(x\right),$$
$$v(x,0) = \tilde{\varepsilon} v_0\left(x\right),\ v_t\left(x,0\right) = \tilde{\varepsilon} v_1\left(x\right), 0 < \tilde{\varepsilon} \ll 1. \tag{2.1}$$

We shall search for the system (1.3) solution in the form $u = \tilde{\varepsilon}\tilde{u}$, $v = \tilde{\varepsilon}\tilde{v}$. Substituting u, v in (1.3) and selecting the terms of $\tilde{\varepsilon}$ power-series expansions, we obtain:

$$\tilde{\varepsilon}\left(\rho A \tilde{u}_{tt} - EA\tilde{u}_{xx}\right) - \tilde{\varepsilon}^2(EA - T)$$
$$\times \left(\tilde{v}_x\tilde{v}_{xx} + \tilde{\varepsilon}\left(\tilde{u}_{xx}\tilde{v}_x^2 - 2\tilde{u}_x\tilde{v}_x\tilde{v}_{xx} + O\left(\tilde{\varepsilon}\right)\right)\right) = 0,$$
$$\tilde{\varepsilon}\left(\rho A \tilde{v}_{tt} - T\tilde{v}_{xx}\right) - \tilde{\varepsilon}^2(EA - T)\left(\tilde{u}_{xx}\tilde{v}_x + \tilde{u}_x\tilde{v}_{xx}\right.$$
$$\left. - \tilde{\varepsilon}\left(\tilde{u}_x^2\tilde{v}_{xx} - \frac{3}{2}\tilde{v}_{xx}\tilde{v}_x^2 + 2\tilde{u}_x\tilde{v}_x\tilde{u}_{xx} + O\left(\tilde{\varepsilon}\right)\right)\right) = 0. \tag{2.2}$$

Let us note that

$$a^2 = E/\rho, \quad b^2 = T/(\rho A), \quad \varepsilon = \tilde{\varepsilon}(b^2 - a^2).$$

By omitting members of the $O\left(\tilde{\varepsilon}^3\right)$ order and neglecting the sign $(\tilde{\ })$, we receive the asymptotic integration problem

$$\begin{cases} u_{tt} - a^2 u_{xx} = -\varepsilon v_x v_{xx}, \\ v_{tt} - b^2 v_{xx} = -\varepsilon(v_x u_{xx} + u_x v_{xx}) \end{cases} \tag{2.3}$$

with the initial conditions corresponding to (2.1). When in system (2.3) $\varepsilon = 0$, we obtain a couple of two independent travelling waves (see, e. g., [30]):

$$u(x,t) = \frac{1}{2}\left(u_0\left(x + at\right) + u_0\left(x - at\right)\right) + \frac{1}{2a}\int_{x-at}^{x+at} u_1\left(\xi\right)\, d\xi,$$

$$v(x,t) = \frac{1}{2}\left(v_0\left(x + bt\right) + v_0\left(x - bt\right)\right) + \frac{1}{2b}\int_{x-bt}^{x+bt} v_1\left(\xi\right)\, d\xi. \tag{2.4}$$

Functions (2.4) are obtained also from the linear model (1.1). However, when $\varepsilon > 0$, function (2.4) will be close to the solution of the system (2.3), in the general case only in a short time interval $t \ll O\left(\varepsilon^{-1}\right)$.

The system (2.1)–(2.3) has a classical (continuously differentiable with respect to the variables t ir x) solution $u(t,x;\varepsilon)$, $v(t,x;\varepsilon)$ in the time interval $t \in \left[0, \frac{\tau^0}{\varepsilon}\right]$, here τ^0 is a constant. We are constructing an asymptotic solution $U(t,x;\varepsilon)$, $V(t,x;\varepsilon)$ uniformly valid in the indicated interval: $\forall \mu > 0 \; \exists \varepsilon_\mu > 0$ such, that for all $\varepsilon \in (0, \varepsilon_\mu]$ is true next estimation

$$\max_{t \in \left[0, \frac{\tau^0}{\varepsilon}\right], x \in R} |U(t,x;\varepsilon) - u(t,x;\varepsilon)| + |V(t,x;\varepsilon) - v(t,x;\varepsilon)| < \mu. \tag{2.5}$$

At a certain moment of time $t = \frac{\tau^1}{\varepsilon}$, there may appear the discontinuous derivatives u_t, u_x, v_t, v_x (gradient catastrophe) is constructed and its asymptotic approximation does not describe the solution. This means that the time interval under analysis $t \in \left[0, \frac{\tau^0}{\varepsilon}\right]$ and $\tau^0 < \tau^1$. Estimation (2.5) is an analog of the well known in asymptotic analysis N.N. Bogoliubov's theorem [24].

The aspects of mathematical substantiation of asymptotic approximations constructed in the paper have been analyzed [13, 14].

3 Construction of the uniformly valid asymptotics

Let us rewrite system (2.3) in the Riemann invariants r_j of the linear part of the system. Note that the asymptotic method does not use the quasi-linear system invariants which may be constructed by the methods of [9] [10].

$$r_1 = u_t - au_x, \quad r_2 = u_t + au_x, \quad r_3 = v_t - bv_x, \quad r_4 = v_t + bv_x.$$

Then we obtain

$$\begin{cases} \dfrac{\partial r_j}{\partial t} \pm a\dfrac{\partial r_j}{\partial x} = -\dfrac{\varepsilon}{4b^2}\left(r_4 - r_3\right)\left(r_{4x} - r_{3x}\right), \quad j = 1, 2, \\[2ex] \dfrac{\partial r_j}{\partial t} \pm b\dfrac{\partial r_j}{\partial x} = -\dfrac{\varepsilon}{4ab}\left(\left(r_4 - r_3\right)\left(r_{2x} - r_{1x}\right)\right. \\[1ex] \qquad\qquad\qquad \left. + \left(r_2 - r_1\right)\left(r_{4x} - r_{3x}\right)\right), \quad j = 3, 4. \end{cases} \tag{3.1}$$

We shall analyze system (3.1) when functions (2.1) are 2π-periodic:

$$r_j\left(0, x; \varepsilon\right) = \sum_{k_j=-\infty}^{+\infty} r_{jk_j}^{(0)} e^{\mathbf{i}k_j x}. \tag{3.2}$$

If to looking for the asymptotic solution of (3.1) as a regular small parameter ε power series expansion

$$r_j(t, x; \varepsilon) = r_{j0}(t, x) + \varepsilon r_{j1}(t, x) + O(\varepsilon^2), \qquad (3.3)$$

we shall see that in the general case (3.3) we will have the secular terms εt (see, e. g., [26]). Therefore, approximation (3.3) may be used only when the values εt are small, i. e. $t \ll \varepsilon^{-1}$, or in a short time t interval. On the other hand, equations (3.1) have a classical (diferentiated) solution when $t \in [0, O(\varepsilon^{-1})]$, i. e. in a long time interval.

Let us note that in papers [5, 6] concerning the model (1.3) the author analyzed the asymptotics, but they had secular terms.

It is a non-trivial problem to construct an asymptotic solution uniformly valid in a long time interval $t \in [0, O(\varepsilon^{-1})]$. Such asymptotics

$$r_j(x, t; \varepsilon) = R_j(\tau, y_j) + O(\varepsilon)$$

is found as a solution of the following system averaged along the characteristics [15, 16]:

$$
\begin{cases}
\dfrac{\partial R_j}{\partial \tau} = -\dfrac{1}{4b^2} \left\langle (R_4(\tau, y_4) - R_3(\tau, y_3)) \right. \\
\qquad \left. \left(\dfrac{\partial R_4(\tau, y_4)}{\partial y_4} - \dfrac{\partial R_3(\tau, y_3)}{\partial y_3} \right) \right\rangle_j , \; j = 1, 2, \\[2mm]
\dfrac{\partial R_3}{\partial \tau} = -\dfrac{1}{4ab} \left\langle \left(R_4(\tau, y_4) \left(\dfrac{\partial R_2(\tau, y_2)}{\partial y_2} - \dfrac{\partial R_1(\tau, y_1)}{\partial y_1} \right) + \right. \right. \\
\qquad \left. \left. (R_2(\tau, y_2) - R_1(\tau, y_1)) \dfrac{\partial R_4(\tau, y_4)}{\partial y_4} \right) \right\rangle_3 , \\[2mm]
\dfrac{\partial R_4}{\partial \tau} = \dfrac{1}{4ab} \left\langle \left(R_3(\tau, y_3) \left(\dfrac{\partial R_2(\tau, y_2)}{\partial y_2} - \dfrac{\partial R_1(\tau, y_1)}{\partial y_1} \right) + \right. \right. \\
\qquad \left. \left. (R_2(\tau, y_2) - R_1(\tau, y_1)) \dfrac{\partial R_3(\tau, y_3)}{\partial y_3} \right) \right\rangle_4 .
\end{cases} \qquad (3.4)
$$

Here, $\tau = \varepsilon t$ is a slow time, $y_1 = x - at$, $y_2 = x + at$, $y_3 = x - bt$, $y_4 = x + bt$ are the fast characteristic variables, $y = (y_1, y_2, y_3, y_4)$. The operators $\langle \cdots \rangle_j$ averaging along characteristics are described as follows:

$$\langle f(\tau, y) \rangle_1 =$$
$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\tau, y_1, y_1 + (\lambda_1 - \lambda_2) s, y_1 + (\lambda_1 - \lambda_3) s, y_1 + (\lambda_1 - \lambda_4) s) \, ds,$$
$$\langle f(\tau, y) \rangle_2 =$$
$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\tau, y_2 + (\lambda_2 - \lambda_1) s, y_2, y_2 + (\lambda_2 - \lambda_3) s, y_2 + (\lambda_2 - \lambda_4) s) \, ds,$$
$$\langle f(\tau, y) \rangle_3 =$$
$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\tau, y_3 + (\lambda_3 - \lambda_1) s, y_3 + (\lambda_3 - \lambda_2) s, y_3, y_3 + (\lambda_3 - \lambda_4) s) \, ds,$$
$$\langle f(\tau, y) \rangle_4 =$$
$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\tau, y_4 + (\lambda_4 - \lambda_1) s, y_4 + (\lambda_4 - \lambda_2) s, y_4 + (\lambda_4 - \lambda_3) s, y_4) \, ds.$$

Here, $\lambda_1 = a$, $\lambda_2 = -a$, $\lambda_3 = b$, $\lambda_4 = -b$.

When all numbers $\frac{\lambda_i - \lambda_j}{\lambda_k - \lambda_j}$ are irrational, the right sides of system (3.4) will be equal to zero and the solutions are expressed by formulas (2.4), i. e. travelling by independent waves at the velocity λ_j. In the opposite case, the right sides of system (3.4) may not equal zero, and in this case the solution of this system depends on the slow time τ. In this paper, we shall limit ourselves to the case when a and b are integer numbers.

We shall obtain an integral differential system which we integrate from 0 to 2π. Sometimes such averaging is called *internal* [15], and this is its essential difference as compared with various partial derivatives averaging schemes [23,24] which may be called the *external* averaging. The systems obtained by internal averaging often remain without a subsequent study and left in the literature as a certain theoretical result of the asymptotic analysis [3,28]. Let us note that system (3.4) does not directly depend on the small parameter ε and thus has no problems of asymptotic integration. Therefore, the problem may be successfully solved by numerical methods [15,16]. Let us note some papers [11,20,27,29] in which, like in our work, the method of two scales and averaging methods are applied, however, the Fourier analysis is performed without directly constructing the averaging system. In our paper, for the definite equations (1.3), we have applied the general method of averaging along characteristics the weakly nonlinear hyperbolic systems. The constructed averaging system allows finding uniformly valid asymptotic approximations of a polynomial form by applying the methods of our earlier work [17]. On the other hand, the theoretical analysis of the obtained averaged system (see [18,19]) allows determining the conditions of appearing combinatoric resonances.

4 Analysis of the averaged system

We shall analyze the solution of system (3.4) by the Fourier series:

$$R_j\left(\tau, y_j\right) = \sum_{k_j=-\infty}^{+\infty} R_{jk_j}\left(\tau\right) e^{\mathbf{i}k_j y_j}. \tag{4.1}$$

The functions $R_{jk_j}\left(\tau\right)$ (3.2) satisfy the initial conditions $R_{jk_j}\left(0\right) = r_{jk_j}^{(0)}$. When the right sides of system (3.4) are equal to zero, correspondingly the functions $R_{jk_j}\left(\tau\right) \equiv r_{jk_j}^{(0)}$, i. e. do not depend on τ. Otherwise we shall have the *resonant wave interaction* and the functions $R_{jk_j}\left(\tau\right)$ will describe the slowly changing wave amplitude.

Let us denote the sets of resonant harmonics:

$$\mathcal{R}_j = \left\{ k_j \in Z \setminus \{0\} : \ \frac{dR_{jk_j}\left(\tau\right)}{d\tau} \not\equiv 0 \right\}.$$

The resonant wave interaction in the system does not appear if $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}_3 = \mathcal{R}_4 = \emptyset$. The sets $\mathcal{R}_j$ are interdependent and are empty only in case when the parameters of the problem $a = \sqrt{\frac{E}{\rho}}$ and $b = \sqrt{\frac{T}{\rho A}}$ meet certain requirements. Suppose that the sets $\mathcal{R}_3$ and $\mathcal{R}_4$ have no such harmonics k_3, k_4

for whom

$$k_3 \left(a \mp b \right) = k_4 \left(a \pm b \right) \tag{4.2}$$

is valid. In this case, we shall have $\mathcal{R}_1 = \mathcal{R}_2 = \emptyset$. On the other hand, the sets $\mathcal{R}_{3,4}$ will be empty, if they and the sets $\mathcal{R}_{1,2}$ have no such harmonics for which

$$k_{1,2} \left(2b \right) = k_{3,4} \left(a \pm b \right) \tag{4.3}$$

is valid.

Note also that for resonances to appear the following relations among all harmonics depending to the set $\mathcal{R}_j$ are necessary:

$$k_{1,2} = k_3 + k_4, \ k_{3,4} = k_{4,3} + k_{1,2}. \tag{4.4}$$

Let us note that the relations (4.3), (4.4) imply that the corresponding coefficients of the Fourier $R_{jk_j} \left(\tau \right)$ should not be equal to zero.

5   Numerical experiments

In practice, we limit ourselves to the first Fourier lines (4.1) harmonics $|k_j| \le K$ and look for the functions $R_{jk_j} \left(\tau \right)$ by the following approximations:

$$R_{jk_j} \left(\tau \right) \approx R_{jk_j}^{(0)} + R_{jk_j}^{(1)} \tau + R_{jk_j}^{(2)} \tau^2 + \cdots + R_{jk_j}^{(N)} \tau^N. \tag{5.1}$$

We find the polynomial coefficients by the method of undetermined coefficients. The same results we obtain by successive iterations.

5.1   Example 1

We shall analyze the problem when $a = 1$, $b = 3$ and the initial conditions are as follows:

$$R_1 \left(0, y_1 \right) = R_2 \left(0, y_2 \right) = 0,$$
$$R_3 \left(0, y_3 \right) = \sin 2y_3 + \sin 4y_3 + 2 \cos 2y_3 + \cos 4y_3,$$
$$R_4 \left(0, y_4 \right) = 3 \sin 2y_4 + \sin 4y_4 + 5 \cos 2y_4 + \cos 4y_4.$$

We shall present the calculated data using the computer algebra system Maple coefficients (5.1):

$$R_1 \left(\tau, y_1 \right) = \left(\frac{2}{3} \tau - \frac{11}{216} \tau^3 + \frac{227}{311040} \tau^5 + \frac{12463}{313528320} \tau^7 \right) \cos 6y_1$$

$$+ \left(-\frac{1}{6} \tau + \frac{11}{864} \tau^3 - \frac{227}{1244160} \tau^5 + \frac{12463}{1254113280} \tau^7 \right) \sin 6y_1,$$

$$R_2 \left(\tau, y_2 \right) = \left(\frac{1}{4} \tau - \frac{1}{432} \tau^3 - \frac{11}{622080} \tau^5 + \frac{23}{67184640} \tau^7 \right) \cos 6y_2$$

$$+ \left(-\frac{1}{12} \tau + \frac{1}{1296} \tau^3 + \frac{11}{1866240} \tau^5 - \frac{23}{201553920} \tau^7 \right) \sin 6y_2,$$

$$R_3\left(\tau, y_3\right) = \left(2 + \frac{1}{72}\tau^2 - \frac{19}{62208}\tau^4 + \frac{181}{134369280}\tau^6\right)\cos 2y_3$$

$$+ \left(1 + \frac{1}{144}\tau^2 - \frac{19}{124416}\tau^4 + \frac{181}{268738560}\tau^6\right)\sin 2y_3$$

$$+ \left(1 - \frac{17}{72}\tau^2 + \frac{85}{10368}\tau^4 + \frac{17}{414720}\tau^6\right)\cos 4y_3$$

$$+ \left(1 - \frac{17}{72}\tau^2 + \frac{85}{10368}\tau^4 + \frac{17}{414720}\tau^6\right)\sin 4y_3,$$

$$R_4\left(\tau, y_4\right) = \left(5 + \frac{5}{144}\tau^2 - \frac{25}{4608}\tau^4 + \frac{1889}{5971968}\tau^6\right)\cos 2y_4$$

$$+ \left(3 + \frac{1}{48}\tau^2 - \frac{5}{1536}\tau^4 + \frac{1889}{9953280}\tau^6\right)\sin 2y_4$$

$$+ \left(1 - \frac{5}{144}\tau^2 + \frac{5}{124416}\tau^4 + \frac{179}{53747712}\tau^6\right)\cos 4y_4$$

$$+ \left(1 - \frac{5}{144}\tau^2 + \frac{5}{124416}\tau^4 + \frac{179}{53747712}\tau^6\right)\sin 4y_4.$$

We see that the sets $\mathcal{R}_{1,2} = \{-6, 6\}$, $\mathcal{R}_{3,4} = \{-4, -2, 2, 4\}$. Thus, conditions (4.2), (4.3) are satisfied: $2k_3 = 4k_4$, $6k_{3,4} = 2k_{1,2}$ or $6k_{3,4} = 4k_{1,2}$. Besides, all harmonics satisfy conditions (4.4): $\pm 6 = \pm 4 \pm 2$, $\pm 4 = \pm 6 \mp 2$ and $\pm 2 = \mp 4 \pm 6$.
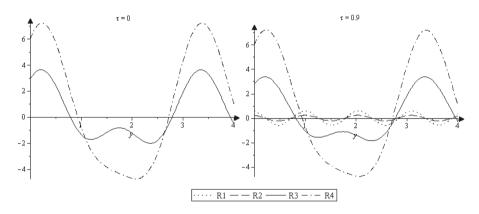

Figure 1. Graphs of example 1 functions R_1, R_2, R_3, R_4

Figure 1 presents graphs of the functions $R_j\left(\tau, y\right)$, $j = 1, 2, 3, 4$ when $\tau = 0$ and $\tau = 0.9$, and $y_1 = y_2 = y_3 = y_4 = y$ are taken. At the initial time moment $\tau = 0$, the functions R_1 and R_2 are absent, and at $\tau = 0.9$ there appear waves, their amplitudes depending on τ. The resonant interactions of waves cause also the dependence of the amplitudes of waves R_3, R_4 on the slow time τ. The amplitudes slowly change, however, at $\tau = 0.9$ in the graph the new waves R_3, R_4 are clearly seen.

5.2   Example 2

Let us take the same values as in Fig. 1: $a = 1$, $b = 3$, and in the initial conditions let us take the non-zero functions R_1, R_2:

$$R_1\,(0, y_1) = \sin 3y_1 + 5\cos 3y_1,$$
$$R_2\,(0, y_2) = 4\sin 3y_2 + 8\cos 3y_2,$$
$$R_3\,(0, y_3) = \sin 2y_3 + \sin 4y_3 + 2\cos 2y_3 + \cos 4y_3,$$
$$R_4\,(0, y_4) = 3\sin 2y_4 + \sin 4y_4 + 5\cos 2y_4 + \cos 4y_4.$$

The calculated coefficients of polynoms (5.1) are presented in Table 1.

Table 1.   The polynomial coefficients at τ in a degree.

		τ^0	τ^1	τ^2	τ^3	τ^4	τ^5
R_1	$\cos 3y_1$	5		0.0217		-0.0005	
	$\sin 3y_1$	1		0.0043		-0.0001	
	$\cos 6y_1$		0.6667		-0.0818		0.0080
	$\sin 6y_1$		-0.1667		0.0204		-0.0020
R_2	$\cos 3y_2$	8		0.2361		-0.0196	
	$\sin 3y_2$	4		0.1181		-0.0098	
	$\cos 6y_2$		0.25		-0.0061		0.0001
	$\sin 6y_2$		-0.0833		0.0020		$0.394 \cdot 10^{-6}$
R_3	$\cos y_3$		0.1667		-0.0057		-0.0002
	$\sin y_3$		2.1667		-0.0740		-0.0032
	$\cos 2y_3$	2		0.0764		-0.0001	
	$\sin 2y_3$	1		-0.0382		$-4.823 \cdot 10^{-6}$	
	$\cos 4y_3$	1		-0.2361		0.0301	
	$\sin 4y_3$	1		-0.2361		0.0301	
R_4	$\cos y_4$		0.1250		-0.0014		$-0.803 \cdot 10^{-7}$
	$\sin y_4$		0.4583		-0.0052		$-0.294 \cdot 10^{-6}$
	$\cos 2y_4$	5		-0.6597		-0.0046	
	$\sin 2y_4$	3		-0.3958		-0.0028	
	$\cos 4y_4$	1		-0.0347		0.0011	
	$\sin 4y_4$	1		-0.0347		0.0011	

We see that the sets $\mathcal{R}_j$ in this case have more elements:

$$\mathcal{R}_{1,2} = \{-6, -3, 3, 6\}, \quad \mathcal{R}_{3,4} = \{-4, -2, -1, 1, 2, 4\}.$$

This means that not only the transverse oscillations induced the longitudinal ones as in example 1, but also the longitudinal oscillations induced a resonance (not only amplitude) effect on the transverse ones. This effect caused the first harmonic – the basic tone – of the transverse oscillations.

The constructed approximations of the Riemann invariants allow to write down the approximations of the required functions u, v, i.e. wave profile approximations:

$$u(x, t; \varepsilon) \approx \frac{1}{2a} \int \left(R_2(\varepsilon t, x + at) - R_1(\varepsilon t, x + at) \right)\, dx$$

$$= \left(1.3333 + 0.0394\,(\varepsilon t)^2 - 0.0033\,(\varepsilon t)^4 \right) \sin 3(x + t)$$

$$- \left(0.6667 + 0.0197\,(\varepsilon t)^2 - 0.0016\,(\varepsilon t)^4\right)\cos 3(x+t)$$

$$- \left(0.8333 + 0.0036\,(\varepsilon t)^2 - 0.0001\,(\varepsilon t)^4\right)\sin 3(x-t)$$

$$+ \left(0.1667 + 0.0007\,(\varepsilon t)^2\right)\cos 3(x-t)$$

$$+ \left(0.0208\,(\varepsilon t) - 0.0005\,(\varepsilon t)^3\right)\sin 6(x+t)$$

$$+ \left(0.0069\,(\varepsilon t) - 0.0002\,(\varepsilon t)^3\right)\cos 6(x+t)$$

$$- \left(0.0556\,(\varepsilon t) + 0.0068\,(\varepsilon t)^3 + 0.0007\,(\varepsilon t)^5\right)\sin 6(x-t)$$

$$- \left(0.0139\,(\varepsilon t) + 0.0017\,(\varepsilon t)^3 + 0.0002\,(\varepsilon t)^5\right)\cos 6(x-t),$$

$$v(x,t;\varepsilon) \approx \frac{1}{2b}\int \left(R_4(\varepsilon t, x+bt) - R_3(\varepsilon t, x+bt)\right)\,dx$$

$$= \left(0.0208\,(\varepsilon t) - 0,0002\,(\varepsilon t)^3\right)\sin(x+3t)$$

$$- \left(0.0764\,(\varepsilon t) - 0.0009\,(\varepsilon t)^3\right)\cos(x+3t)$$

$$- \left(0.0278\,(\varepsilon t) - 0,0009\,(\varepsilon t)^3\right)\sin(x-3t)$$

$$+ \left(0.3611\,(\varepsilon t) - 0.0123\,(\varepsilon t)^3 - 0.0005\,(\varepsilon t)^5\right)\cos(x-3t)$$

$$+ \left(0.4167 - 0.0550\,(\varepsilon t)^2 - 0.0004\,(\varepsilon t)^4 + 0.0002\,(\varepsilon t)^6\right)\sin 2(x+3t)$$

$$+ \left(0.2500 + 0.0330\,(\varepsilon t)^2 + 0.0002\,(\varepsilon t)^4 - 0.0001\,(\varepsilon t)^6\right)\cos 2(x+3t)$$

$$- \left(0.1667 - 0.0064\,(\varepsilon t)^2\right)\sin 2(x-3t)$$

$$+ \left(0.0833 - 0.0032\,(\varepsilon t)^2\right)\cos 2(x-3t)$$

$$+ \left(0.0417 - 0.0014\,(\varepsilon t)^2\right)\sin 4(x+3t)$$

$$- \left(0.04167 - 0.0014\,(\varepsilon t)^2\right)\cos 4(x+3t)$$

$$- \left(0.0417 - 0.0098\,(\varepsilon t)^2 - 0.0013\,(\varepsilon t)^4 + 0.0001\,(\varepsilon t)^6\right)\sin 4(x-3t)$$

$$+ \left(0.0417 - 0.0098\,(\varepsilon t)^2 + 0.0013\,(\varepsilon t)^4 - 0.0001\,(\varepsilon t)^6\right)\cos 4(x-3t).$$

The Figures 2–5 present the graphs of the functions u and v at different moments of the slow time $\tau = \varepsilon t$. Each graph contains three values of the fast time t: $t = 10\tau$, $t = 100\tau$, $t = 1000\tau$.

6 Conclusions

The system constructed by means of internal averaging along characteristics allows finding asymptotic solutions, elucidating their properties, determining the resonance-appearing conditions.
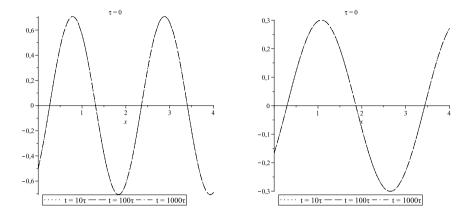
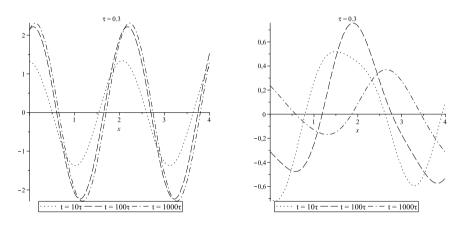**Figure 2.** Profiles of waves u, v at time moment $\tau = 0$

Figure 3. Profiles of waves u, v at time moment $\tau = 0.3$

The small-amplitude periodic oscillations in the Carrier system for a non-resonance case are described by independent waves travelling to different sides with the velocities $a = \sqrt{\frac{E}{\rho}}$ by transverse and $b = \sqrt{\frac{T}{\rho A}}$ by longitudinal waves.

In the resonance case, wave amplitudes depend on the slow time τ, and the periodic functions describing the waves have the harmonics which have been absent at the initial moment of time. The resonant interaction of transverse and longitudinal waves may appear only if relations (4.2), (4.3) and (4.4) among the harmonics and the velocities a, b are valid.

If at the initial moment of time the transverse oscillations contain no harmonics satisfying the relation (4.2), no resonance appears in the system. This means that the longitudinal oscillations alone cannot arouse the transversal ones, but the transverse oscillations may induce the longitudinal ones in the presence of (4.2).

In this case, the longitudinal oscillations, together with the transversal, ones may induce new harmonics of the transverse vibrations (overtones) of (4.2).
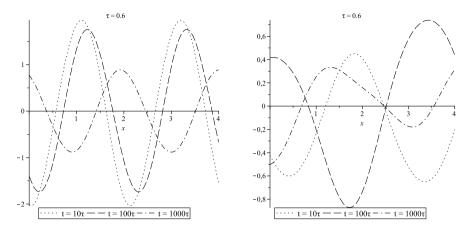
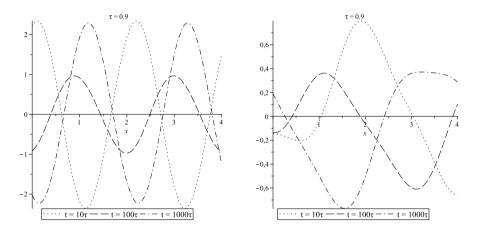**Figure 4.**   Profiles of waves u, v at time moment $\tau = 0.6$

Figure 5.   Profiles of waves u, v at time moment $\tau = 0.9$

Note that all conclusions are valid for the first asymptotic approximations (2.2). By presenting in the system (2.2) terms of the higher order of the small parameter ε, we may construct the second asymptotic approximation and analyze the harmonics which appear not only because of combinatoric resonances (4.2),(4.3),(4.4) but also because of the system's nonlinearities. It is obvious that these effects will be of a higher ε order, i. e. they will be weaker.

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