

Proximal Point Algorithm for a Common of Countable Families of Inverse Strongly Accretive Operators and Nonexpansive Mappings with Convergence Analysis

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Abstract. In this paper, we investigate and analyze a proximal point algorithm via viscosity approximation method with error. This algorithm is introduced for finding a common zero point for a countable family of inverse strongly accretive operators and a countable family of nonexpansive mappings in Banach spaces. Our result can be extended to some well known results from a Hilbert space to a uniformly convex and 2-uniformly smooth Banach space. Finally, we establish the strong convergence theorems for the proximal point algorithm. Also, some illustrative numerical examples are presented.

Keywords: Proximal Point Algorithm; Zero point; inverse strongly accretive operator; nonexpansive mapping.

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1 Introduction

The problems of finding a zero point for monotone operators play an important role in modern optimization and analysis. It can be related to many kinds of important problems, such as convex minimization problems, equilibrium problems, variational inequality problems and others. In order to approximate the solution to these problems, many authors have intensively studied the convergence of such problems in several setting spaces. Martinet [15] first introduced the *proximal point algorithm (PPA)* in a Hilbert space H ; for starting $x_0 \in H$, a sequence $\{x_n\}$ is generated by

$$x_{n+1} = J_{r_n}^A(x_n), \forall n \in \mathbb{N},$$

where $J_{r_n}^A = (I + r_n A)^{-1}$ is the resolvent operator of A with I being the identity mapping and $\{r_n\} \subset (0, \infty)$ is a regularization sequence.

Later, Rockafellar [18] has studied the proximal point algorithm which generates a sequence $\{x_n\}$ according to the following algorithm:

$$x_{n+1} = J_{r_n}^A x_n + e_n, \forall n \in \mathbb{N},$$

where $\{e_n\}$ is a sequence of errors and $\{r_n\} \subset (0, \infty)$ is a sequence of regularization parameters. Rockafellar proved that if $A^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a solution of a zero point of A .

In 2000, Moudafi [16] introduced the viscosity approximation method for finding fixed point of a nonexpansive mapping S in a Hilbert space; for given $x_0 \in C$, the sequence is defined by the following algorithm:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) Sx_n, \forall n \geq 0,$$

where $f : C \rightarrow C$ is a contraction mapping and $\{\alpha_n\} \subseteq (0, 1)$ satisfies some condition. This sequence converges strongly to a fixed point of S .

After that, Xu [20] developed the viscosity approximation method in both Hilbert and Banach spaces. Strong convergence theorems for zero points of monotone operators were established in a Banach space. Moreover, many authors constructed some approximation algorithms for some nonlinear variational inclusions in Hilbert spaces or Banach spaces by using the resolvent operator technique. In [6, 13, 14], the *PPA* was extended to the case of sum of two monotone operators.

In 2014, Eslamian [8] proposed Rockafellar's proximal point algorithm for finding a zero point of a finite family of monotone operators via viscosity method in a Hilbert space. Eslamian [8] proposed the following algorithm:

$$x_{n+1} = a_{n,0} f(x_n) + a_{n,1} J_{r_n}^{T_1} x_n + a_{n,2} J_{r_n}^{T_2} x_n + \dots + a_{n,m} J_{r_n}^{T_m} x_n + e_n$$

for all $n \geq 1$ where $\sum_{i=0}^m a_{n,i} = 1$ and for each $i = 1, 2, \dots, m$, T_i , are a finite family of monotone operators of H . If $\{a_{n,i}\}$, $\{e_n\}$ and $\{r_n\}$ are positive sequences and satisfy some conditions, then the sequence $\{x_n\}$ converges strongly.

Nowadays, several iterative methods have been proposed and analyzed to find a common solution of two different fixed point problems, such as a fixed

point problem of nonexpansive mapping, fixed point problem of variational (inclusions) inequalities and other. However, there are a few iterative methods for finding a common solution of two different infinite countable family of the fixed point problems.

In this paper, motivated by [8, 16], we are interested in the problem for finding a common solution of two different fixed point problems which are a common element of a zero point for an infinite countable family of β -inverse strongly accretive operators and a common fixed point of an infinite countable nonexpansive mappings in uniformly convex and 2-uniformly smooth Banach space. Our main result study these problems in uniformly convex and 2-uniformly smooth Banach space such as the spaces L_p , l_p or Sobolev space W_m^p (where $p \geq 2$), which is more general than Hilbert space. In addition, our main theorem solves the problems for finding a common element of an infinite family of accretive operators, which is more general than the problem of finding a common element of a finite family of monotone operators. Consequently, the strong convergence theorem is obtained. Finally, we have studied the convergence analysis of the proximal point algorithm. Also, some illustrative numerical examples (using Matlab software) are presented.

2 Preliminaries

Let E be a real Banach space and E^* be the dual space of E . Let $\langle \cdot, \cdot \rangle$ be the pairing between E and E^* . For all $x \in E$ and $x^* \in E^*$, let the value of x^* at x be denoted by $\langle x, x^* \rangle$. The *normalized duality mapping* $J : E \rightarrow 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\|\}, \quad \forall x \in E.$$

A *single-value* normalized duality mapping is denoted by j , which means that a mapping $j : E \rightarrow E^*$ such that, for all $u \in E$, $j(u) \in E^*$ satisfies the following:

$$\langle j(u), u \rangle = \|j(u)\| \|u\|, \quad \|j(u)\| = \|u\|.$$

If $E = H$ is a Hilbert space, then $J = I$, where I is identity mapping. If E is a *smooth Banach space*, then J is single-valued.

A Banach space E is said to satisfy *Opial's condition* if for each sequence $\{x_n\}_{n=0}^\infty$ in E such that $\{x_n\}$ converges weakly to some x in E , the inequality

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$$

holds for all $y \in E$ with $y \neq x$. In fact, a Banach space with a weakly sequentially continuous duality mapping has the Opial's condition; see [9]. We know that if E admits a weakly sequentially continuous duality mapping, then E is smooth. It is well known that all Hilbert spaces and l_p , ($p > 1$) satisfy Opial's condition, while L_p does not satisfy this condition except $p = 2$.

A Banach space E is called a *strictly convex* if it satisfies the following condition

$$\|x\| = \|y\| = \|(1 - \lambda)x + \lambda y\|, \quad \forall x, y \in E \text{ and } 0 < \lambda < 1 \implies x = y.$$

Let $S(E) = \{x \in E : \|x\| = 1\}$ be the unit sphere of E . E is said to be *uniformly convex* if for each $\epsilon \in (0, 2]$ there exists a constant $\delta = \delta(\epsilon) > 0$ such that for all $x, y \in S(E)$, if $\|x - y\| \geq \epsilon$ then $\left\| \frac{x+y}{2} \right\| \leq 1 - \delta$. It is well known that uniformly convex is strictly convex.

A Banach space E is said to be *smooth* if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S(E)$. In this case, the norm $\|\cdot\|$ of E is said to be *Gâteaux differentiable norm*.

The norm of E is said to be *uniformly Gâteaux differentiable norm* if for each $y \in S(E)$, the limit (2.1) is attained uniformly for all $x \in S(E)$ and it is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit (2.1) is attained uniformly for all $y \in S(E)$. Moreover, it is said to be *uniformly smooth* if the limit (2.1) is attained uniformly for all $(x, y) \in S(E) \times S(E)$. It is well known that if the norm of E is smooth, then the duality mapping J is single-valued and norm to weak* continuous on a bounded subset of E .

The *modulus of smoothness* of E is the function $\rho : [0, \infty) \rightarrow [0, \infty)$ defined by $\rho(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = t\}$. A Banach space E is an *uniformly smooth* if and only if $\lim_{t \rightarrow 0} \frac{\rho(t)}{t} = 0$.

A Banach space E is said to be *q-uniformly smooth* if for $1 < q \leq 2$ be a fixed real number, there exists a constant $c > 0$ such that $\rho(t) \leq ct^q$ for all $t > 0$. In the case $q = 2$, E is said to be *2-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(t) \leq ct^2$ for all $t > 0$. Typical examples of both uniformly convex and 2-uniformly smooth Banach space are $l_p, L_p, (p \geq 2)$, the Sobolev space $W_m^p, (p \geq 2)$ and all Hilbert space.

Recall that every uniformly smooth space is smooth [5]. If E is a *q-uniformly smooth*, then E is uniformly smooth. Hence *2-uniformly smooth* of Banach space is a uniformly smooth and it is *uniformly Gâteaux differentiable norms*. If E is *uniformly Gâteaux differentiable norms*, then the duality mapping $J : E \rightarrow 2^{E^*}$ is a single-valued and J is norm to weak* uniformly continuous on a bounded subset of E .

Let C be a nonempty closed convex subset of a Banach space E and $D \subset C$, then a mapping $Q : C \rightarrow D$ is said to be *sunny* if $Q(x + t(x - Q(x))) = Q(x)$ whenever $Qx + t(x - Q(x)) \in C$ for all $x \in C$ and $t \geq 0$.

A mapping $Q : C \rightarrow C$ is called a *retraction* if $Q^2 = Q$. Note that if a mapping Q is a retraction, then $Qz = z$ for all $z \in R(Q)$ where $R(Q)$ is the range of Q . A subset D of Q is called a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction from C onto D .

Lemma 1. [17] *Let E be a smooth Banach space and let C be a nonempty subset of E . Let $Q : E \rightarrow C$ be a retraction and let J be the normalized duality mapping on E . Then, the following statements are equivalent:*

- (i) Q is sunny and nonexpansive;
- (ii) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy) \rangle, \forall x, y \in E$;
- (iii) $\|(x - y) - (Qx - Qy)\|^2 \leq \|x - y\|^2 - \|Qx - Qy\|^2$; (iv) $\langle x - Qx, J(y - Qx) \rangle \leq 0, \forall x \in E, y \in C$.

Lemma 2. [11] *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let S be a nonexpansive mapping of C into itself with $Fix(S) \neq \emptyset$. Then, the set $Fix(S)$ is a sunny nonexpansive retract of C .*

In 2004, Xu [20] studied the following continuous scheme:

$$x_t = tf(x_t) + (1 - t)Sx_t,$$

where f is a k -contraction and S is nonexpansive mapping. On a uniformly smooth Banach space, Xu [20] proved that the sequence $\{x_t\}$ converges strongly to a fixed point of S . If we defines mapping $Q : \Pi_c \rightarrow Fix(S)$, where Π_c denote the set of k -contraction, by $Q(f) := \lim_{t \rightarrow \infty} x_t$, then $Q(f)$ solves the following variational inequality:

$$\langle (I - f)Q(f), j(Q(f) - p) \rangle \leq 0, \forall p \in Fix(S).$$

It well known that if $E = H$ is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection from E onto C , that is $Q_C = P_C$. Let C be a nonempty closed convex subset of a smooth Banach space E and let $x \in E$ and let $x_0 \in C$. Then we have from Lemma 1 that $x_0 = Q_C x$ if and only if $\langle x - x_0, J(y - x_0) \rangle \leq 0$ for all $y \in C$, where Q_C is a sunny nonexpansive retraction from E onto C .

Let C be a nonempty closed convex subset of E . Recall the following definitions:

1. Let $f : C \rightarrow E$ be an operator. Then, T is called *k-contraction* if there exists a coefficient k ($0 < k < 1$) such that

$$\|fx - fy\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

2. Let $S : C \rightarrow E$ be an operator. Then, S is called *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

3. Let $A : C \rightarrow E$ be an operator. Then, A is called *accretive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0, \quad \forall x, y \in C.$$

4. Let $A : C \rightarrow E$ be an operator. Then, A is called *β -inverse-strongly accretive* if there exists a constant $\beta > 0$ and $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \beta \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

An accretive operator A is said to be maximal accretive if there is no proper accretive extension of A . If A is an m -accretive operator then A is said to be maximal accretive [7]. In a real Hilbert space, an accretive operator is called *monotone*. Moreover, A is called *maximal monotone* if $R(I + rA) = H$. In this paper, we use $A^{-1}(0)$ to denote the set of zeros of A . For an accretive operator A , we can define a nonexpansive single-valued mapping $J_r^A : R(I + rA) \rightarrow D(A)$ by $J_r^A = (I + rA)^{-1}$ for each $r > 0$, which is called the resolvent of A . It is known that $0 \in A(x) \Leftrightarrow x \in Fix(J_r^A)$.

In the sequel to give our main results, we need the following lemmas.

Lemma 3. [4] Let C be a closed convex subset of a strictly convex Banach space E . Let $T_i : C \rightarrow C$ be nonexpansive mappings for each $i \geq 1$ such that $\bigcap_{i=1}^{\infty} \text{Fix}(T_i)$ is nonempty. Let $\{\alpha_i\}$ be a real number sequence in $(0, 1)$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Then, the mapping \acute{S} on C defined by $\acute{S}x = \sum_{i=1}^{\infty} \alpha_i T_i x$ for all $x \in C$ is well defined. Moreover, the mapping \acute{S} is nonexpansive mapping and $\text{Fix}(\acute{S}) = \bigcap_{i=1}^{\infty} \text{Fix}(T_i)$.

Remark 1. For each $n \geq 1$, if we set $T_i = S_n J_{r_n}^{A_i}$ then T_i is nonexpansive mappings for all $i \geq 1$. Indeed, since S_n and $J_{r_n}^{A_i}$ are nonexpansive mappings, then we obtain that

$$\begin{aligned} \|T_i x - T_i y\| &= \|S_n J_{r_n}^{A_i} x - S_n J_{r_n}^{A_i} y\|, \quad \forall n \in \mathbb{N} \\ &\leq \|J_{r_n}^{A_i} x - J_{r_n}^{A_i} y\|, \quad \forall i \in \mathbb{N} \leq \|x - y\|. \end{aligned}$$

Thus, $T_i = S_n J_{r_n}^{A_i}$ is a nonexpansive mapping for each $n \geq 1$.

Lemma 4. [19] Let $q \in (1, \infty)$, $\lambda \in [0, 1]$ and $W_q(\lambda) := \lambda^q(1 - \lambda) + \lambda(1 - \lambda)^q$. Let E be a real q -uniformly smooth Banach space. Then, there exist constants $c_q > 0$ such that for all $x, y \in E$, the following inequality holds:

$$\|\lambda x + (1 - \lambda)y\|^q \leq \lambda \|x\|^q + (1 - \lambda) \|y\|^q - W_q(\lambda) c_q \|x - y\|^q.$$

Lemma 5. Let E be a 2-uniformly smooth of a real Banach space. Then, for each $x_i \in E$, $\alpha_i \in [0, 1]$, $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} \alpha_i = 1$, we have

$$\left\| \sum_{i=1}^{\infty} \alpha_i x_i \right\|^2 \leq \sum_{i=1}^{\infty} \alpha_i \|x_i\|^2. \quad (2.2)$$

Proof. We will prove this lemma by the mathematical induction.

(1) If $k = 2$, by using Lemma 4, we have

$$\|\alpha_1 x_1 + \alpha_2 x_2\|^2 \leq \alpha_1 \|x_1\|^2 + (1 - \alpha_1) \|x_2\|^2.$$

Hence, the conclusion holds.

(2) Suppose that the inequality (2.2) holds for $k = n - 1$, i.e.,

$$\left\| \sum_{i=1}^{n-1} \alpha_i x_i \right\|^2 \leq \sum_{i=1}^{n-1} \alpha_i \|x_i\|^2.$$

We want to show that the inequality holds for $k = n$. Let $\alpha_n \neq 1$ be chosen in such a way that $\sum_{i=1}^{\infty} \alpha_i = 1$. It follows from the induction hypotheses that

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i x_i \right\|^2 &= \left\| \sum_{i=1}^{n-1} \alpha_i x_i + \alpha_n x_n \right\|^2 \\ &= \left\| (1 - \alpha_n) \frac{\sum_{i=1}^{n-1} \alpha_i x_i}{(1 - \alpha_n)} + \alpha_n x_n \right\|^2 \\ &\leq (1 - \alpha_n) \left\| \frac{\sum_{i=1}^{n-1} \alpha_i x_i}{(1 - \alpha_n)} \right\|^2 + \alpha_n \|x_n\|^2 \\ &\leq \sum_{i=1}^{n-1} \alpha_i \|x_i\|^2 + \alpha_n \|x_n\|^2 \leq \sum_{i=1}^n \alpha_i \|x_i\|^2. \end{aligned}$$

This completes the proof. \square

Lemma 6. [19] *Let E be a 2-uniformly smooth Banach space. Then, for any $x, y, \in E$, the following inequality holds*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x) \rangle + 2\|Ky\|^2, \quad j(x) \in J(x),$$

where K is the 2-uniformly smooth constant of E .

Lemma 7. [10] *Let E be an real Banach space and J the normalized duality map on E . Then, for any $x, y, \in E$, the following inequality holds;*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \forall j(x + y) \in J(x + y).$$

Lemma 8. *Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space E with the 2-uniformly smooth constant K . Let A be an β -inverse strongly accretive operator of C into E with a constant $\beta > 0$. Then, we have*

$$\|J_r^A x - J_r^A y\|^2 \leq \|x - y\|^2 - 2\left(\frac{\beta}{r} - K^2\right)\|(x - J_r^A x) - (y - J_r^A y)\|^2.$$

In particular, if $r \in (0, \frac{\beta}{K^2})$, then J_r^A is a nonexpansive.

Proof. Let $x, y \in C$. By using Lemma 6, we obtain that

$$\begin{aligned} \|J_r^A x - J_r^A y\|^2 &= \|(x - y) - (x - y) + J_r^A x - J_r^A y\|^2 \\ &= \|(x - y) - ((x - J_r^A x) - (y - J_r^A y))\|^2 \\ &= \|(x - y) - r(Ax - Ay)\|^2 \\ &\leq \|x - y\|^2 - 2r\langle Ax - Ay, j(x - y) \rangle + 2K^2 r^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2r\beta \|Ax - Ay\|^2 + 2K^2 r^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2r(\beta - K^2 r) \|Ax - Ay\|^2. \end{aligned}$$

Moreover, if $r \in (0, \frac{\beta}{K^2})$, we obtain that

$$\begin{aligned} \|J_r^A x - J_r^A y\|^2 &\leq \|x - y\|^2 - 2r(\alpha - K^2 r) \left\| \frac{1}{r}(x - J_r^A x) - \frac{1}{r}(y - J_r^A y) \right\|^2 \\ &\leq \|x - y\|^2 - 2r(\beta - K^2 r) \frac{1}{r^2} \|(x - J_r^A x) - (y - J_r^A y)\|^2 \\ &\leq \|x - y\|^2 - 2\left(\frac{\beta}{r} - K^2\right)\|(x - J_r^A x) - (y - J_r^A y)\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

Thus, if $r \in (0, \frac{\beta}{K^2})$, then J_r^A is a nonexpansive mapping. \square

Lemma 9. [2] *(The Resolvent Identity) For all $r > 0, s > 0$ and $x \in E$, then*

$$J_r^B x = J_s^B \left(\frac{s}{r} x + \left(1 - \frac{s}{r}\right) J_r^B x \right).$$

Lemma 10. [3] (*Demiclosed principle*) Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $S : C \rightarrow C$ be a nonexpansive mapping. Then $I - S$ is demiclosed at zero, i.e., $x_n \rightarrow x$ and $x_n - Sx_n \rightarrow 0$ imply $x = Sx$.

Lemma 11. [1] Let C be a nonempty closed convex subset of a Banach space E and let $\{S_0, S_1, S_2, \dots\}$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}x - S_nx\| : x \in C\} < \infty$. Then, for each $y \in C$, $\{S_ny\}$ strongly converges to some point of C . Moreover, if S is a mapping of C into itself defined by $Sy = \lim_{n \rightarrow \infty} S_ny$, $\forall y \in C$. Then, $\limsup_{n \rightarrow \infty} \{\|Sx - S_nx\| : x \in C\} = 0$.

Lemma 12. [12] Assume that $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the condition

$$a_{n+1} \leq (1 - t_n)a_n + t_nb_n + c_n, \forall n \geq 0,$$

where $\{t_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} t_n = 0$ and $\sum_{n \rightarrow \infty} t_n = \infty$, $\{b_n\}$ is a sequence such that $\limsup_{n \rightarrow \infty} b_n \leq 0$ and $\{c_n\}$ is a positive number sequence such that $\sum_{n \rightarrow \infty} c_n < \infty$. Then, $\lim_{n \rightarrow \infty} a_n = 0$.

3 Main Result

Theorem 1. Let E be a uniformly convex and 2-uniformly smooth of real Banach space and let C be a nonempty closed convex subset of E which has a weakly continuous duality mapping J from E to E^* . For any $i \in \mathbb{N}$, let $\{A_i\}_{i=1}^{\infty}$ be an infinite sequence of β_i -inverse strongly accretive operators in E such that $\beta := \inf_{i \geq 1} \{\beta_i\} > 0$ and $\bigcap_{i=1}^{\infty} \overline{D(A_i)} \subseteq C \subseteq \bigcap_{i=1}^{\infty} R(I + rA_i)$ for all $r > 0$. Let $\{S_n\}_{n=1}^{\infty}$ be an infinite sequence of nonexpansive mapping from C into itself such that $F := (\bigcap_{n=1}^{\infty} Fix(S_n)) \cap (\bigcap_{i=1}^{\infty} A_i^{-1}(0)) \neq \emptyset^1$. Let $f : C \rightarrow C$ be a contraction mapping with the constant $k \in (0, 1)$ and $J_{r_n}^{A_i} = (I + r_nA_i)^{-1}$ be a resolvent of A_i for each $r_n > 0$. For given $x_1 \in \bigcap_{i=1}^{\infty} \overline{D(A_i)}$, let $\{x_n\}$ be a sequence defined by the following:

$$x_{n+1} = \alpha_{n,0}f(x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}S_nJ_{r_n}^{A_i}x_n + e_n, \quad \forall n \geq 1, \tag{3.1}$$

where $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$, $\{\alpha_{n,i}\} \subset (0, 1)$ for all $i \geq 0$, $\{r_n\} \subset (0, \frac{\beta}{K^2})$ where K is the 2-uniformly smooth constant and $\{e_n\} \subset (0, \infty)$. Suppose that these sequences satisfy the following conditions:

- (a) $\lim_{n \rightarrow \infty} \alpha_{n,0} = 0$, $\sum_{n=1}^{\infty} \alpha_{n,0} = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1,0} - \alpha_{n,0}| < \infty$;
- (b) for each $i \geq 1$, $\lim_{n \rightarrow \infty} \alpha_{n,i} = \alpha_i \in (0, 1)$;
- (c) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ and $\lim_{n \rightarrow \infty} r_n = r > 0$;

¹ For example, we set $S_nx = \frac{x}{n+1}$ for all $n \geq 1$ and $A_ix = ix$ for all $i \geq 1$. Note the $F \neq \emptyset$ because $0 \in \bigcap_{n=1}^{\infty} Fix(S_n)$ and $0 \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$.

(d) $\sum_{n=1}^{\infty} e_n < \infty$.

Assume that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}x - S_nx\| : x \in B\} < \infty$ for any bounded subset B of C . Let S be a mapping of C into itself defined by $Sx = \lim_{n \rightarrow \infty} S_nx$ for all $x \in C$ such that $\text{Fix}(S) = \bigcap_{n=1}^{\infty} \text{Fix}(S_n)$. Then, the sequence $\{x_n\}$ strongly converges to a point $p \in F$, where $p = Q_F f(p)$ and $Q_F f$ is a sunny nonexpansive retraction from E onto F .

Proof. **Step 1.** We want to show that $\{x_n\}$ is bounded. Let $p \in F$. Then we have $p \in \text{Fix}(S_n)$ for all $n \geq 1$ and $p \in A_i^{-1}(0)$ for all $i \geq 1$. Since each resolvent $J_{r_n}^{A_i}$ is a nonexpansive mapping for all $i \geq 1$ and $n \geq 1$, we obtain that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_{n,0}f(x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}S_nJ_{r_n}^{A_i}x_n + e_n - p\| \\ &\leq \|\alpha_{n,0}f(x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}S_nJ_{r_n}^{A_i}x_n - p\| + e_n \\ &\leq \alpha_{n,0}\|f(x_n) - p\| + \sum_{i=1}^{\infty} \alpha_{n,i}\|S_nJ_{r_n}^{A_i}x_n - S_np\| + e_n \\ &\leq \alpha_{n,0}(\|f(x_n) - f(p)\| + \|f(p) - p\|) + \sum_{i=1}^{\infty} \alpha_{n,i}\|J_{r_n}^{A_i}x_n - J_{r_n}^{A_i}p\| + e_n \\ &\leq \alpha_{n,0}(\|f(x_n) - f(p)\| + \|f(p) - p\|) + \sum_{i=1}^{\infty} \alpha_{n,i}\|x_n - p\| + e_n \\ &\leq \alpha_{n,0}k\|x_n - p\| + (1 - \alpha_{n,0})\|x_n - p\| + \alpha_{n,0}\|f(p) - p\| + e_n \\ &= [\alpha_{n,0}k + (1 - \alpha_{n,0})]\|x_n - p\| + \alpha_{n,0}\|f(p) - p\| + e_n \\ &= [1 - \alpha_{n,0}(1 - k)]\|x_n - p\| + \alpha_{n,0}\|f(p) - p\| + e_n \\ &= [1 - \lambda_n]\|x_n - p\| + \lambda_n \frac{\|f(p) - p\|}{(1 - k)} + e_n, \end{aligned}$$

where $\lambda_n := \alpha_{n,0}(1 - k)$. Then, it follows that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \max \left\{ \|x_n - p\|, \frac{\|f(p) - p\|}{1 - k} \right\} + e_n \\ &\leq \max \left\{ \|x_{n-1} - p\|, \frac{\|f(p) - p\|}{1 - k} \right\} + e_{n-1} + e_n \\ &\leq \dots \leq \max \left\{ \|x_0 - p\|, \frac{\|f(p) - p\|}{1 - k} \right\} + \sum_{n=1}^{\infty} e_n. \end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded. Also, we obtain that $\{f(x_n)\}$, $\{S_nJ_{r_n}^{A_i}x_n\}$, and $\{J_{r_n}^{A_i}x_n\}$ are bounded for all $i \geq 1$ because $S_nJ_{r_n}^{A_i}$ and $J_{r_n}^{A_i}$ are nonexpansive mappings for all $i \geq 1$ and f is a contraction mapping.

Step 2. We will show that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. This observes that

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|(\alpha_{n+1,0}f(x_{n+1}) + \sum_{i=1}^{\infty} \alpha_{n+1,i}S_{n+1}J_{r_{n+1}}^{A_i}x_{n+1} + e_{n+1}) \\
&\quad - (\alpha_{n,0}f(x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}S_nJ_{r_n}^{A_i}x_n + e_n)\| \\
&\leq \|\alpha_{n+1,0}f(x_{n+1}) - \alpha_{n,0}f(x_n) - \alpha_{n+1,0}f(x_n) + \alpha_{n+1,0}f(x_n)\| \\
&\quad + |e_{n+1} - e_n| + \|\sum_{i=1}^{\infty} \alpha_{n+1,i}S_{n+1}J_{r_{n+1}}^{A_i}x_{n+1} - \sum_{i=1}^{\infty} \alpha_{n+1,i}S_nJ_{r_n}^{A_i}x_n \\
&\quad + \sum_{i=1}^{\infty} \alpha_{n+1,i}S_nJ_{r_n}^{A_i}x_n - \sum_{i=1}^{\infty} \alpha_{n,i}S_nJ_{r_n}^{A_i}x_n\| \\
&= \alpha_{n+1,0}\|f(x_{n+1}) - f(x_n)\| + |\alpha_{n+1,0} - \alpha_{n+1,0}| \|f(x_n)\| + e_{n+1} + e_n \\
&\quad + \sum_{i=1}^{\infty} \alpha_{n+1,i}\|S_{n+1}J_{r_{n+1}}^{A_i}x_{n+1} - S_nJ_{r_n}^{A_i}x_n\| \\
&\quad + \|\sum_{i=1}^{\infty} \alpha_{n+1,i}S_nJ_{r_n}^{A_i}x_n - \sum_{i=1}^{\infty} \alpha_{n,i}S_nJ_{r_n}^{A_i}x_n\| \\
&= \alpha_{n+1,0}k\|x_{n+1} - x_n\| + |\alpha_{n+1,0} - \alpha_{n+1,0}| \|f(x_n)\| + e_{n+1} + e_n \\
&\quad + \sum_{i=1}^{\infty} \alpha_{n+1,i}\|S_{n+1}J_{r_{n+1}}^{A_i}x_{n+1} - S_nJ_{r_n}^{A_i}x_n\| \\
&\quad + \|\sum_{i=1}^{\infty} \alpha_{n+1,i}S_nJ_{r_n}^{A_i}x_n - \sum_{i=1}^{\infty} \alpha_{n,i}S_nJ_{r_n}^{A_i}x_n\|. \tag{3.2}
\end{aligned}$$

For each $i \geq 1$, by using Lemma 9, we have the following estimate

$$\begin{aligned}
&\|S_{n+1}J_{r_{n+1}}^{A_i}x_{n+1} - S_nJ_{r_n}^{A_i}x_n\| \\
&= \|S_{n+1}J_{r_{n+1}}^{A_i}x_{n+1} - S_{n+1}J_{r_n}^{A_i}x_n + S_{n+1}J_{r_n}^{A_i}x_n - S_nJ_{r_n}^{A_i}x_n\| \\
&\leq \|J_{r_{n+1}}^{A_i}x_{n+1} - J_{r_n}^{A_i}x_n\| + \|S_{n+1}J_{r_n}^{A_i}x_n - S_nJ_{r_n}^{A_i}x_n\| \\
&\leq \|J_{r_{n+1}}^{A_i}x_{n+1} - J_{r_{n+1}}^{A_i}x_n + J_{r_{n+1}}^{A_i}x_n - J_{r_n}^{A_i}x_n\| + \|S_{n+1}J_{r_n}^{A_i}x_n - S_nJ_{r_n}^{A_i}x_n\| \\
&\leq \|x_{n+1} - x_n\| + \|J_{r_n}^{A_i}(\frac{r_n}{r_{n+1}}x_n - (1 - \frac{r_n}{r_{n+1}})J_{r_{n+1}}^{A_i}x_n) - J_{r_n}^{A_i}x_n\| \\
&\quad + \|S_{n+1}J_{r_n}^{A_i}x_n - S_nJ_{r_n}^{A_i}x_n\| \\
&\leq \|x_{n+1} - x_n\| + |\frac{r_{n+1} - r_n}{r_{n+1}}| \|J_{r_{n+1}}^{A_i}x_n - x_n\| + \|S_{n+1}J_{r_n}^{A_i}x_n - S_nJ_{r_n}^{A_i}x_n\|.
\end{aligned}$$

Substituting it into the inequality (3.2), we obtain that

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &\leq \alpha_{n+1,0}k\|x_{n+1} - x_n\| + |\alpha_{n+1,0} - \alpha_{n+1,0}| \|f(x_n)\| + e_{n+1} + e_n \\
&\quad + \|\sum_{i=1}^{\infty} \alpha_{n+1,i}S_nJ_{r_n}^{A_i}x_n - \sum_{i=1}^{\infty} \alpha_{n,i}S_nJ_{r_n}^{A_i}x_n\| + \sum_{i=1}^{\infty} \alpha_{n+1,i}
\end{aligned}$$

$$\begin{aligned}
 & \{ \|x_{n+1} - x_n\| + | \frac{r_{n+1} - r_n}{r_{n+1}} | \|J_{r_{n+1}}^{A_i} x_n - x_n\| + \|S_{n+1} J_{r_n}^{A_i} x_n - S_n J_{r_n}^{A_i} x_n\| \} \\
 & = \alpha_{n+1,0} k \|x_{n+1} - x_n\| + |\alpha_{n+1,0} - \alpha_{n+1,0}| \|f(x_n)\| + e_{n+1} + e_n \\
 & + \| \sum_{i=1}^{\infty} \alpha_{n+1,i} S_n J_{r_n}^{A_i} x_n - \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n \| + \sum_{i=1}^{\infty} \alpha_{n+1,i} \|x_{n+1} - x_n\| \\
 & + \sum_{i=1}^{\infty} \alpha_{n+1,i} | \frac{r_{n+1} - r_n}{r_{n+1}} | \|J_{r_{n+1}}^{A_i} x_n - x_n\| \\
 & + \sum_{i=1}^{\infty} \alpha_{n+1,i} \|S_{n+1} J_{r_n}^{A_i} x_n - S_n J_{r_n}^{A_i} x_n\| \\
 & = (1 - \alpha_{n+1,0}(1 - k)) \|x_{n+1} - x_n\| + |\alpha_{n+1,0} - \alpha_{n+1,0}| \|f(x_n)\| + e_{n+1} + e_n \\
 & + \| \sum_{i=1}^{\infty} \alpha_{n+1,i} S_n J_{r_n}^{A_i} x_n - \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n \| + \sum_{i=1}^{\infty} \alpha_{n+1,i} | \frac{r_{n+1} - r_n}{r_{n+1}} | \\
 & \quad \times \|J_{r_{n+1}}^{A_i} x_n - x_n\| + \sum_{i=1}^{\infty} \alpha_{n+1,i} \|S_{n+1} J_{r_n}^{A_i} x_n - S_n J_{r_n}^{A_i} x_n\| \\
 & = (1 - \alpha_{n+1,0}(1 - k)) \|x_{n+1} - x_n\| + c_n, \tag{3.3}
 \end{aligned}$$

where

$$\begin{aligned}
 c_n & := |\alpha_{n+1,0} - \alpha_{n+1,0}| \|f(x_n)\| + e_{n+1} + e_n \\
 & + \| \sum_{i=1}^{\infty} \alpha_{n+1,i} S_n J_{r_n}^{A_i} x_n - \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n \| + \sum_{i=1}^{\infty} \alpha_{n+1,i} | \frac{r_{n+1} - r_n}{r_{n+1}} | \\
 & \quad \times \|J_{r_{n+1}}^{A_i} x_n - x_n\| + \sum_{i=1}^{\infty} \alpha_{n+1,i} \|S_{n+1} J_{r_n}^{A_i} x_n - S_n J_{r_n}^{A_i} x_n\|.
 \end{aligned}$$

Since $\{x_n\}$, $\{J_{r_n}^{A_i} x_n\}$ and $\{S_n J_{r_n}^{A_i} x_n\}$ are bounded sequences for all $i \geq 1$, then we can find the positive real numbers L_i and M_i such that

$$\|S_n J_{r_n}^{A_i} x_n\| \leq L_i \leq \max_{i \geq 1} \{L_i\} := L, \quad \|J_{r_n}^{A_i} x_n - x_n\| \leq M_i \leq \max_{i \geq 1} \{M_i\} := M.$$

Since $\lim_{n \rightarrow \infty} r_n = r > 0$, thus $\lim_{n \rightarrow \infty} \frac{1}{r_n}$ exists and so there is a positive real number r' such that $|\frac{1}{r_n}| < r'$ for all $n \in \mathbb{N}$. Then, we obtain that

$$\begin{aligned}
 & \sum_{n=1}^{\infty} (\| \sum_{i=1}^{\infty} \alpha_{n+1,i} S_n J_{r_n}^{A_i} x_n - \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n \|) \\
 & \leq \sum_{n=1}^{\infty} (\| \sum_{i=1}^{\infty} \alpha_{n+1,i} - \sum_{i=1}^{\infty} \alpha_{n,i} \| \|S_n J_{r_n}^{A_i} x_n\|) \\
 & \leq L \sum_{n=1}^{\infty} \| \sum_{i=1}^{\infty} \alpha_{n+1,i} - \sum_{i=1}^{\infty} \alpha_{n,i} \| \\
 & = L \sum_{n=1}^{\infty} |(1 - \alpha_{n+1,0}) - (1 - \alpha_{n,0})| = L \sum_{n=1}^{\infty} |\alpha_{n+1,0} - \alpha_{n,0}| < \infty \tag{3.4}
 \end{aligned}$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_{n+1,i} \left| \frac{r_{n+1}-r_n}{r_{n+1}} \right| \|J_{r_{n+1}}^{A_i} x_n - x_n\| \right) &\leq \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_{n+1,i} r' M |r_{n+1}-r_n| \right) \\ &= r' M \sum_{n=1}^{\infty} |r_{n+1} - r_n| \left(\sum_{i=1}^{\infty} \alpha_{n+1,i} \right) < \infty. \end{aligned} \tag{3.5}$$

By the boundness of $\{J_{r_n}^{A_i} x_n\}$ and our assumptions, we observe that

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_{n+1,i} \|S_{n+1} J_{r_n}^{A_i} x_n - S_n J_{r_n}^{A_i} x_n\| \right) &\leq \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_{n+1,i} \sup\{\|S_{n+1}z - S_n z\| : z \in \{J_{r_n}^{A_i} x_n\}\} \right) \\ &= \sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in \{J_{r_n}^{A_i} x_n\}\} \left(\sum_{i=1}^{\infty} \alpha_{n+1,i} \right) < \infty. \end{aligned} \tag{3.6}$$

Since $\sum_{n=1}^{\infty} |\alpha_{n+1,0} - \alpha_{n,0}| < \infty$, $\sum_{n=1}^{\infty} e_n < \infty$ and $\{f(x_n)\}$ is bounded, by (3.4), (3.5) and (3.6), then we see that the series $\sum_{n=1}^{\infty} c_n < \infty$. By using Lemma 12 and (3.3), we conclude that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

Step 3. We will show that $\lim_{n \rightarrow \infty} \|x_n - J_r^{A_i} x_n\| = 0$ for each $i \geq 1$ and $r > 0$ where $r = \lim_{n \rightarrow \infty} r_n$. By using Lemma 5 and Lemma 8, we obtain that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_{n,0} f(x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n + e_n - p\|^2 \\ &= \|(\alpha_{n,0} f(x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n - p) + e_n\|^2 \\ &= \|\alpha_{n,0} f(x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n - p\|^2 \\ &\quad + 2e_n \|\alpha_{n,0} f(x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n - p\| + e_n^2 \\ &= \|\alpha_{n,0} f(x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n - p\|^2 + g_n \\ &= \|\alpha_{n,0} (f(x_n) - p) + \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n - \sum_{i=1}^{\infty} \alpha_{n,i} p\|^2 + g_n \\ &\leq \alpha_{n,0} \|f(x_n) - p\|^2 + \sum_{i=1}^{\infty} \alpha_{n,i} \|J_{r_n}^{A_i} x_n - J_{r_n}^{A_i} p\|^2 + g_n \end{aligned}$$

$$\begin{aligned}
 &\leq \alpha_{n,0} \|f(x_n) - p\|^2 + \sum_{i=1}^{\infty} \alpha_{n,i} \{ \|x_n - p\|^2 \\
 &\quad - 2(\frac{\beta}{r_n} - K^2) \|(x_n - J_{r_n}^{A_i} x_n) - (p - J_{r_n}^{A_i} p)\|^2 \} + g_n \\
 &= \alpha_{n,0} \|f(x_n) - p\|^2 + (1 - \alpha_{n,0}) \|x_n - p\|^2 \\
 &\quad - \sum_{i=1}^{\infty} 2\alpha_{n,i} (\frac{\beta}{r_n} - K^2) \|(x_n - J_{r_n}^{A_i} x_n)\|^2 + g_n, \tag{3.7}
 \end{aligned}$$

where $g_n = 2e_n \|\alpha_{n,0} f(x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n - p\| + e_n^2$. Since $\{f(x_n)\}$ and $\{S_n J_{r_n}^{A_i} x_n\}$ are bounded sequences, $\sum_{i=1}^{\infty} \alpha_{n,i} < 1$ and $\lim_{n \rightarrow \infty} e_n = 0$, we have $\lim_{n \rightarrow \infty} g_n = 0$. From (3.7), we get that

$$\begin{aligned}
 &\sum_{i=1}^{\infty} 2\alpha_{n,i} (\frac{\beta}{r_n} - K^2) \|(x_n - J_{r_n}^{A_i} x_n)\|^2 \\
 &\leq \alpha_{n,0} \|f(x_n) - p\|^2 + (1 - \alpha_{n,0}) \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + g_n \\
 &\leq (1 - \alpha_{n,0}) \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_{n,0} \|f(x_n) - p\|^2 + g_n \\
 &\leq \|x_n - x_{n+1}\| \|x_n - p + x_{n+1} - p\| + \alpha_{n,0} \|f(x_n) - p\|^2 + g_n. \tag{3.8}
 \end{aligned}$$

Let us fix $i \in \mathbb{N}$. Since $\alpha_{n,i} \subset (0, 1)$ for all $i \in \mathbb{N}$ and $2(\frac{\beta}{r_n} - K^2) > 0$. Then, by (3.8), we get that

$$\begin{aligned}
 2\alpha_{n,i} (\frac{\beta}{r_n} - K^2) \|x_n - J_{r_n}^{A_i} x_n\|^2 &\leq \sum_{i=1}^{\infty} 2\alpha_{n,i} (\frac{\beta}{r_n} - K^2) \|(x_n - J_{r_n}^{A_i} x_n)\|^2 \\
 &\leq \|x_n - x_{n+1}\| \|x_n - p + x_{n+1} - p\| + \alpha_{n,0} \|f(x_n) - p\|^2 + g_n. \tag{3.9}
 \end{aligned}$$

Since $\{x_n\}$ and $\{f(x_n)\}$ are bounded sequences, $\lim_{n \rightarrow \infty} g_n = 0$, $\lim_{n \rightarrow \infty} \alpha_{n,0}$, $\lim_{n \rightarrow \infty} \alpha_{n,i} = \alpha_i$ for each $i \geq 1$ and by (3.9), we conclude that

$$\lim_{n \rightarrow \infty} \|x_n - J_{r_n}^{A_i} x_n\| = 0. \tag{3.10}$$

By using Lemma 9 we note that

$$\begin{aligned}
 \|x_n - J_r^{A_i} x_n\| &\leq \|x_n - J_{r_n}^{A_i} x_n\| + \|J_{r_n}^{A_i} x_n - J_r^{A_i} x_n\| \\
 &= \|x_n - J_{r_n}^{A_i} x_n\| + \|J_r^{A_i} \left(\frac{r}{r_n} x_n + (1 - \frac{r}{r_n}) J_{r_n}^{A_i} x_n \right) - J_r^{A_i} x_n\| \\
 &\leq \|x_n - J_{r_n}^{A_i} x_n\| + \left\| \left(\frac{r}{r_n} x_n - (1 - \frac{r}{r_n}) J_{r_n}^{A_i} x_n \right) - x_n \right\| \\
 &= \|x_n - J_{r_n}^{A_i} x_n\| + \left| 1 - \frac{r}{r_n} \right| \|J_{r_n}^{A_i} x_n - x_n\| \\
 &\leq \|x_n - J_{r_n}^{A_i} x_n\| + \|J_{r_n}^{A_i} x_n - x_n\| = 2\|x_n - J_{r_n}^{A_i} x_n\|. \tag{3.11}
 \end{aligned}$$

Hence, by (3.10), we observe that

$$\lim_{n \rightarrow \infty} \|x_n - J_r^{A_i} x_n\| = 0. \tag{3.12}$$

Step 4. We will show that there exists a unique point $p \in F$ such that

$$\limsup_{n \rightarrow \infty} \langle f(p) - p, j(x_n - p) \rangle \leq 0, \quad p = Q_F f(p).$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ which converges weakly to \bar{x} . To prove this, we divide to 4 steps.

Step 4.1. We will show that $\bar{x} \in \bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$. Note that

$$\begin{aligned} \|x_{n_m} - J_r^{A_i} \bar{x}\| &\leq \|x_{n_m} - J_r^{A_i} x_{n_m}\| + \|J_r^{A_i} x_{n_m} - J_r^{A_i} \bar{x}\| \\ &\leq \|x_{n_m} - J_r^{A_i} x_{n_m}\| + \|x_{n_m} - \bar{x}\|. \end{aligned}$$

This implies that

$$\limsup_{m \rightarrow \infty} \|x_{n_m} - J_r^{A_i} \bar{x}\| \leq \limsup_{m \rightarrow \infty} \|x_{n_m} - \bar{x}\|. \quad (3.13)$$

Since a smooth Banach space with a weakly sequential continuous duality mapping j has the Opial's property. Then, by using (3.13), we obtain that $J_r^{A_i} \bar{x} = \bar{x}$ for all $i \in \mathbb{N}$. This is complete the proof because

$$\begin{aligned} J_r^{A_i} \bar{x} = \bar{x}, \quad \forall i \in \mathbb{N}, &\iff \bar{x} \in \text{Fix}(J_r^{A_i}), \quad \forall i \in \mathbb{N} \\ &\iff \bar{x} \in A_i^{-1}(0), \quad \forall i \in \mathbb{N} \\ &\iff \bar{x} \in \bigcap_{i=1}^{\infty} A_i^{-1}(0). \end{aligned}$$

Step 4.2. Next, we will show that $\bar{x} \in \text{Fix}(S) = \bigcap_{n=1}^{\infty} \text{Fix}(S_n)$. Note that

$$\|S_n J_{r_n}^{A_i} x_n - J_{r_n}^{A_i} x_n\| \leq \|S_n J_{r_n}^{A_i} x_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - J_{r_n}^{A_i} x_n\|.$$

We observe that

$$\begin{aligned} \|x_{n+1} - S_n J_{r_n}^{A_i} x_n\| &= \|\alpha_{n,0} f(x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n + e_n - S_n J_{r_n}^{A_i} x_n\| \\ &\leq \|\alpha_{n,0} f(x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n - S_n J_{r_n}^{A_i} x_n\| + e_n \\ &= \|\alpha_{n,0} f(x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n - \left(\sum_{i=0}^{\infty} \alpha_{n,i}\right) S_n J_{r_n}^{A_i} x_n\| + e_n \\ &= \|\alpha_{n,0} f(x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n - (\alpha_{n,0} + \sum_{i=1}^{\infty} \alpha_{n,i}) S_n J_{r_n}^{A_i} x_n\| + e_n \\ &\leq \alpha_{n,0} \|f(x_n) - S_n J_{r_n}^{A_i} x_n\| + e_n. \end{aligned}$$

Since $\{f(x_n)\}$ and $\{S_n J_{r_n}^{A_i} x_n\}$ are bounded sequences, by condition (a) and (d), we obtain that

$$\lim_{n \rightarrow \infty} \|S_n J_{r_n}^{A_i} x_n - x_n\| = 0. \quad (3.14)$$

Note that

$$\begin{aligned} \|Sx_n - x_n\| &\leq \|Sx_n - SJ_{r_n}^{A_i}x_n\| + \|SJ_{r_n}^{A_i}x_n - S_nJ_{r_n}^{A_i}x_n\| + \|S_nJ_{r_n}^{A_i}x_n - x_n\| \\ &\leq \|x_n - J_{r_n}^{A_i}x_n\| + \|SJ_{r_n}^{A_i}x_n - S_nJ_{r_n}^{A_i}x_n\| + \|S_nJ_{r_n}^{A_i}x_n - x_n\|. \end{aligned}$$

By using Lemma 11, (3.12) and (3.14), we obtain that

$$\lim_{n \rightarrow \infty} \|Sx_n - x_n\| = 0.$$

Since $x_{n_m} \rightarrow \bar{x}$, by Lemma 10, we have $\bar{x} \in Fix(S) = \bigcap_{n=1}^{\infty} Fix(S_n)$.

Step 4.3. We will show that there exists a unique point $p \in F$ and $p = Q_F f(p)$. Indeed, since $F := (\bigcap_{n=1}^{\infty} Fix(S_n)) \cap (\bigcap_{i=1}^{\infty} A_i^{-1}(0)) \neq \emptyset$ is closed and convex. By using Lemma 2, we obtain that the set F is a sunny nonexpansive retract from E onto F , i.e., Q_F is well defined. We see that $Q_F f$ is contraction of C into itself. In fact, since Q_F is a nonexpansive mapping,

$$\|Q_F(f)(x) - Q_F(f)(y)\| \leq \|f(x) - f(y)\| \leq k\|x - y\|,$$

where $k \in (0, 1)$. By Banach contraction principle, there exists a unique element $p \in F$ such that $p = Q_F f(p)$. That is $\bar{x} = p = Q_F f(p) \in F$.

Step 4.4. We will show that $\limsup_{n \rightarrow \infty} \langle f(p) - p, j(x_n - p) \rangle \leq 0$.
Choose a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle f(p) - p, j(x_n - p) \rangle = \lim_{n \rightarrow \infty} \langle f(p) - p, j(x_{n_m} - p) \rangle.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_{m_j}}\}$ of $\{x_{n_m}\}$ which $x_{n_{m_j}} \rightarrow \bar{x}$. Without loss of generality, we can assume that $x_{n_m} \rightarrow \bar{x}$. It follows that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle f(p) - p, j(x_n - p) \rangle &= \lim_{m \rightarrow \infty} \langle f(p) - p, j(x_{n_m} - p) \rangle \\ &= \langle f(p) - p, j(\bar{x} - p) \rangle \leq 0. \end{aligned}$$

Step 5. Finally, we will show that $x_n \rightarrow p$.

By using Lemma 7 and Lemma 5, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_{n,0}f(x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}S_nJ_{r_n}^{A_i}x_n + e_n - p\|^2 \\ &\leq \left\| \sum_{i=1}^{\infty} \alpha_{n,i}S_nJ_{r_n}^{A_i}x_n - \sum_{i=1}^{\infty} \alpha_{n,i}p + e_n \right\|^2 + 2\alpha_{n,0}\langle f(x_n) - p, j(x_{n+1} - p) \rangle \\ &\leq \left\| \sum_{i=1}^{\infty} \alpha_{n,i}S_nJ_{r_n}^{A_i}x_n - \sum_{i=1}^{\infty} \alpha_{n,i}p \right\|^2 + 2\alpha_{n,0}\langle f(x_n) - p, j(x_{n+1} - p) \rangle \\ &\quad + 2e_n \left\| \sum_{i=1}^{\infty} \alpha_{n,i}S_nJ_{r_n}^{A_i}x_n - \sum_{i=1}^{\infty} \alpha_{n,i}p \right\| + e_n^2 \end{aligned}$$

$$\begin{aligned}
&\leq \left\| \sum_{i=1}^{\infty} \alpha_{n,i} (S_n J_{r_n}^{A_i} x_n - p) \right\|^2 + h_n + 2\alpha_{n,0} \langle f(x_n) - p, j(x_{n+1} - p) \rangle \\
&\leq (1 - \alpha_{n,0})^2 \|x_n - p\|^2 + h_n + 2\alpha_{n,0} \langle f(x_n) - f(p), j(x_{n+1} - p) \rangle \\
&\quad + 2\alpha_{n,0} \langle f(p) - p, j(x_{n+1} - p) \rangle \\
&\leq (1 - \alpha_{n,0})^2 \|x_n - p\|^2 + h_n + 2\alpha_{n,0} \|f(x_n) - f(p)\| \|x_{n+1} - p\| \\
&\quad + 2\alpha_{n,0} \langle f(p) - p, j(x_{n+1} - p) \rangle \\
&\leq (1 - \alpha_{n,0})^2 \|x_n - p\|^2 + h_n + 2\alpha_{n,0} k \|x_n - p\| \|x_{n+1} - p\| \\
&\quad + 2\alpha_{n,0} \langle f(p) - p, j(x_{n+1} - p) \rangle \\
&\leq (1 - \alpha_{n,0})^2 \|x_n - p\|^2 + h_n + \alpha_{n,0} k \{ \|x_n - p\|^2 + \|x_{n+1} - p\|^2 \} \\
&\quad + 2\alpha_{n,0} \langle f(p) - p, j(x_{n+1} - p) \rangle,
\end{aligned}$$

where $h_n = 2e_n \left\| \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n - \sum_{i=1}^{\infty} \alpha_{n,i} p \right\| + e_n^2$. This implies that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \frac{(1 - \alpha_{n,0})^2 + \alpha_{n,0} k}{1 - \alpha_{n,0} k} \|x_n - p\|^2 \\
&\quad + \frac{2\alpha_{n,0}}{1 - \alpha_{n,0} k} \langle f(p) - p, j(x_{n+1} - p) \rangle + \frac{h_n}{(1 - \alpha_{n,0} k)} \\
&= \frac{1 - 2\alpha_{n,0} + \alpha_{n,0} k}{1 - \alpha_{n,0} k} \|x_n - p\|^2 + \frac{\alpha_{n,0}^2}{1 - \alpha_{n,0} k} \|x_n - p\|^2 \\
&\quad + \frac{2\alpha_{n,0}}{1 - \alpha_{n,0} k} \langle f(p) - p, j(x_{n+1} - p) \rangle + \frac{h_n}{(1 - \alpha_{n,0} k)} \\
&\leq \left(1 - \frac{2(1-k)\alpha_{n,0}}{1 - \alpha_{n,0} k}\right) \|x_n - p\|^2 + \frac{2(1-k)\alpha_{n,0}}{1 - \alpha_{n,0} k} \left(\frac{\alpha_{n,0} M}{2(1-k)}\right) \\
&\quad + \frac{1}{1-k} \langle f(p) - p, j(x_{n+1} - p) \rangle + \frac{h_n}{(1 - \alpha_{n,0} k)} \\
&\leq (1 - t_n) \|x_n - p\|^2 + t_n b_n + c'_n,
\end{aligned}$$

where $\tilde{M} = \sup\{\|x_n - p\|^2 : n \geq 0\}$, $t_n = \frac{2(1-k)\alpha_{n,0}}{1 - \alpha_{n,0} k}$, $b_n = \frac{\alpha_{n,0} \tilde{M}}{2(1-k)} + \frac{1}{1-k} \langle f(p) - p, j(x_{n+1} - p) \rangle$, and $c'_n = \frac{h_n}{(1 - \alpha_{n,0} k)}$. By assumptions, we see that $\lim_{n \rightarrow \infty} t_n = 0$, $\sum_{n=1}^{\infty} t_n = \infty$ and $\sum_{n=1}^{\infty} c'_n < \infty$. By step 4.4, so we have $\limsup_{n \rightarrow \infty} b_n \leq 0$. Therefore, by using Lemma 12, we obtain that the sequence $\{x_n\}$ strongly converges to $p = Q_F f(p)$. This completes the proof. \square

Remark 2. We given some examples concerning of the sequence $\{\alpha_{n,0}\}$ and $\{\alpha_{n,i}\}$ that satisfy the control condition of Theorem 1 as follows:

$$(1) \text{ if } \alpha_{n,0} := \frac{1}{n} \text{ and } \alpha_{n,i} := \left(1 - \frac{1}{n}\right) \frac{1}{2^i}, n \geq 1.$$

It easy to see that

$$\lim_{n \rightarrow \infty} \alpha_{n,0} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0, \quad \sum_{n=1}^{\infty} \alpha_{n,0} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

$$\sum_{n=1}^{\infty} |\alpha_{n+1,0} - \alpha_{n,0}| = \sum_{n=1}^{\infty} \left| \frac{1}{n+1} - \frac{1}{n} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

For each $i \geq 1$, we have

$$\lim_{n \rightarrow \infty} \alpha_{n,i} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) \frac{1}{2^i} = \frac{1}{2^i} \in (0, 1).$$

Since $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$, then

$$\begin{aligned} \sum_{i=0}^{\infty} \alpha_{n,i} &= \alpha_{n,0} + \sum_{i=1}^{\infty} \alpha_{n,i} = \frac{1}{n} + \sum_{i=1}^{\infty} \left(1 - \frac{1}{n}\right) \frac{1}{2^i} \\ &= \frac{1}{n} + \left(1 - \frac{1}{n}\right) \sum_{i=1}^{\infty} \frac{1}{2^i} = 1. \end{aligned}$$

- (2) if $\alpha_{n,0} := \frac{1}{n+1}$ and $\alpha_{n,i} := \left(1 - \frac{1}{n+1}\right) \frac{1}{2^i}$, $n \geq 1$.
- (3) if $\alpha_{n,0} := \frac{1}{\sqrt{n}}$ and $\alpha_{n,i} := \left(1 - \frac{1}{\sqrt{n}}\right) \frac{1}{2^i}$, $n \geq 1, i \geq 1$.
- (4) if $\alpha_{n,0} := \frac{1}{n^p}$ and $\alpha_{n,i} := \left(1 - \frac{1}{n^p}\right) \frac{1}{2^i}$, $n \geq 1, i \geq 1, p \leq 1$.

By setting $S_n \equiv I$ for all $n \in \mathbb{N}$ in Theorem 1, we obtain the following Corollary:

Corollary 1. Let E be a uniformly convex and 2-uniformly smooth of real Banach space and let C be a nonempty closed convex subset of E which has a weakly continuous duality mapping J from E to E^* . For any $i \in \mathbb{N}$, let $\{A_i\}_{i=1}^{\infty}$ be an infinite sequence of β_i -inverse strongly accretive operators in E such that $\beta := \inf_{i \geq 1} \{\beta_i\} > 0$ and $\bigcap_{i=1}^{\infty} \overline{D(A_i)} \subset C \subset \bigcap_{i=1}^{\infty} R(I + rA_i)$ for all $r > 0$. Let $f : C \rightarrow C$ be a contraction mapping with a constant $k \in (0, 1)$ and $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$ be a resolvent of A_i for $r_n > 0$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by the following:

$$x_{n+1} = \alpha_{n,0} f(x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} J_{r_n}^{A_i} x_n + e_n, \quad \forall n \in \mathbb{N},$$

where $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$, $\{\alpha_i\} \subset (0, 1)$ for all $i \geq 0$, $\{r_n\} \subset (0, \frac{\beta}{K^2})$ where K is the 2-uniformly smooth constant, and $\{e_n\} \subset (0, \infty)$. Suppose that the following conditions hold:

- (a) $\lim_{n \rightarrow \infty} \alpha_{n,0} = 0$, $\sum_{n=1}^{\infty} \alpha_{n,0} = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1,0} - \alpha_{n,0}| < \infty$;
- (b) for each $i \geq 1$, $\lim_{n \rightarrow \infty} \alpha_{n,i} = \alpha_i \in (0, 1)$;
- (c) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ and $\lim_{n \rightarrow \infty} r_n = r > 0$;
- (d) $\sum_{n=1}^{\infty} e_n < \infty$.

Then, the sequence $\{x_n\}$ strongly converges to a point $p \in F$, where $p = Q_F f(x)$.

If we put $J_{r_n}^{A_i} \equiv I$ for all $i \in \mathbb{N}, n \in \mathbb{N}$ in Theorem 1, we obtain the following Corollary:

Corollary 2. Let E be a uniformly convex and 2-uniformly smooth of real Banach space and let C be a nonempty closed convex subset of E which has a weakly continuous duality mapping j from E to E^* . For any $n \in \mathbb{N}$, let $\{S_n\}_{n=1}^\infty$ be an infinite sequence of nonexpansive mapping from C onto itself such that $F := \bigcap_{n=1}^\infty \text{Fix}(S_n) \neq \emptyset$. Let $f : C \rightarrow C$ be a contraction mapping with a constant $k \in (0, 1)$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by the following:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S_n x_n + e_n, \quad \forall n \geq 1.$$

If sequence $\{\alpha_n\} \subset (0, 1)$ and $\{e_n\} \subset (0, \infty)$ are satisfy the following conditions:

- (a) $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty$ and $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
- (b) $\sum_{n=1}^\infty e_n < \infty$.

Assume that $\sum_{n=1}^\infty \sup\{\|S_{n+1}x - S_nx\| : x \in B\} < \infty$ for any bounded subset B of C . Let S be a mapping of C into itself defined by $Sx = \lim_{n \rightarrow \infty} S_nx$ for all $x \in C$ and suppose that $\text{Fix}(S) = \bigcap_{n=1}^\infty \text{Fix}(S_n)$. Then, the sequence $\{x_n\}$ converges strongly to a point $p \in F$, where $p = Q_F f(x)$ and $Q_F f(x)$ is sunny nonexpansive retraction from E onto F .

It is well known that in a Hilbert space, the 2-uniformly smooth constant $K = 1$. If we set $E = H$ in Theorem 1, we obtain the following Corollary:

Corollary 3. Let C be a nonempty close convex subset of a Hilbert space H . For any $i \in \mathbb{N}$, let $\{A_i\}_{i=1}^\infty$ be an infinite sequence of β_i -inverse strongly accretive operators in H such that $\beta := \inf_{i \geq 1} \{\beta_i\} > 0$ and $\bigcap_{i=1}^\infty \overline{D(A_i)} \subseteq C \subseteq \bigcap_{i=1}^\infty R(I+rA_i)$ for all $r > 0$. Let $\{S_n\}_{n=1}^\infty$ be an infinite sequence of nonexpansive mapping from C onto itself such that $F := \bigcap_{n=1}^\infty \text{Fix}(S_n) \cap \bigcap_{i=1}^\infty A_i^{-1}(0) \neq \emptyset$. Let $f : C \rightarrow C$ be a contraction mapping with a constant $k \in (0, 1)$ and $J_{r_n}^{A_i} = (I+r_nA_i)^{-1}$ be a resolvent of A_i for $r_n > 0$. For given $x_1 \in \bigcap_{i=1}^\infty \overline{D(A_i)}$, let $\{x_n\}$ be a sequence defined by the following:

$$x_{n+1} = \alpha_{n,0} f(x_n) + \sum_{i=1}^\infty \alpha_{n,i} S_n J_{r_n}^{A_i} x_n + e_n, \quad \forall n \geq 1,$$

where $\sum_{i=0}^\infty \alpha_{n,i} = 1$ and sequence $\{\alpha_{n,i}\} \subset (0, 1)$ for $i \geq 0$. If sequence $\{r_n\} \subset (0, \beta)$, and $\{e_n\} \subset (0, \infty)$ are satisfy the following conditions:

- (a) $\lim_{n \rightarrow \infty} \alpha_{n,0} = 0, \sum_{n=1}^\infty \alpha_{n,0} = \infty$ and $\sum_{n=1}^\infty |\alpha_{n+1,0} - \alpha_{n,0}| < \infty$;
- (b) for each $i \geq 1, \lim_{n \rightarrow \infty} \alpha_{n,i} = \alpha_i \in (0, 1)$;
- (c) $\sum_{n=1}^\infty |r_{n+1} - r_n| < \infty$ and $\lim_{n \rightarrow \infty} r_n = r > 0$;

(d) $\sum_{n=1}^{\infty} e_n < \infty$.

Assume that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}x - S_nx\| : x \in B\} < \infty$ for any bounded subset B of C . Let S be a mapping of C into itself defined by $Sx = \lim_{n \rightarrow \infty} S_nx$ for all $x \in C$ and suppose that $Fix(S) = \bigcap_{n=1}^{\infty} Fix(S_n)$. Then, the sequence $\{x_n\}$ strongly converges to a point $p \in F$ and $p = P_F f(x)$ and $P_C f(x)$ is a metric projection from H onto F .

Remark 3. We note that Theorem 1 improves and extends in the following aspects:

- (a) Our result can be extend [8, Theorem 3.1] from a Hilbert space to a uniformly convex and 2-uniformly smooth Banach space.
- (b) Our result can be extend from solving the problems of finding a common element of a finite sequence of monotone operators [8, Theorem 3.1] to solving the problems of finding a common element of an infinite family of inverse-strongly accretive operators.
- (c) The iterative scheme (3.1) in Theorem 1 is different from Theorem 3.1 in [8] because the $J_{r_n}^{A_i}$ is replace by the composite mapping $S_n J_{r_n}^{A_i}$.

4 Numerical examples

Let us show numerical example to demonstrate the performance and convergence of our main result as follows.

Example 1. Let $E = \mathbb{R}$ and $C = [-1, 000, 1, 000]$. Define an infinite sequence of mappings $S_n : C \rightarrow C$ and operators $A_i : \mathbb{R} \rightarrow \mathbb{R}$ by

$$S_n(x) = \frac{x - 1}{n + 1} + 1 \text{ for all } n \in \mathbb{N}$$

and

$$A_i(x) = \frac{i}{i + 1}(x - 1) \text{ for all } i \in \mathbb{N}.$$

It is not hard to see that S_n are nonexpansive mappings for all $n \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \sup\{\|S_{n+1}x - S_nx\| : x \in B\} < \infty$$

for any bounded subset B of C and A_i are β_i -inverse strongly accretive operators where $\beta_i = 1 + \frac{1}{i}$ for all $i \in \mathbb{N}$. So $\beta := \inf_{i \geq 1} \{\beta_i\} = 1 > 0$. Moreover, we see that the resolvent of each A_i is

$$J_{r_n}^{A_i}(x) = (I + r_n A_i)^{-1}(x) = \frac{(1 + i)x + r_n i}{1 + i + r_n i} \text{ for } r_n > 0.$$

Note that the resolvent $J_{r_n}^{A_i}$ are nonexpansive mappings for fixed $i \in \mathbb{N}$.

By the definition of S_n and A_i , we observe that $1 \in F$. The Theorem 1 certifies that the sequence x_n in Algorithm 1 converges to the point in F . That is $x_n \rightarrow 1$.

Suppose that a contraction mapping $f : C \rightarrow C$ is defined by

$$f(x) = \frac{x+1}{2} \quad \text{for all } x \in C$$

and two sequences are defined by

$$\alpha_{n,i} = \begin{cases} \frac{1}{n+1}, & i = 0, \\ \left(1 - \frac{1}{n+1}\right) \frac{1}{2^i}, & i \geq 1, \end{cases} \quad \{r_n\} = \left\{ \frac{n+1}{2n} \right\}, \quad n \geq 0, i \geq 1.$$

It is easy to see that these sequences satisfy all of conditions in Theorem 1. Now, the following algorithm is presented for finding a proximal point $x \in F$ as in Theorem 1.

Algorithm 1 Proximal point algorithm via viscosity approximation method

Step 1. Choose $x_0 \in C$ arbitrarily and let $n = 0$.

Step 2. Compute the series value given by

$$y_n = \sum_{i=1}^{\infty} \alpha_{n,i} S_n J_{r_n}^{A_i} x_n.$$

Step 3. Update $x_{n+1} = \alpha_{n,0} f(x_n) + y_n$.

Step 4. Put $n := n + 1$ and return to Step 1.

In this experiment, we first test the effect of initial points on solution by choosing three initial points $x_0 = 50, 10, -20$. The behavior of x_n for Algorithm 1 is indicated in Figure 1.

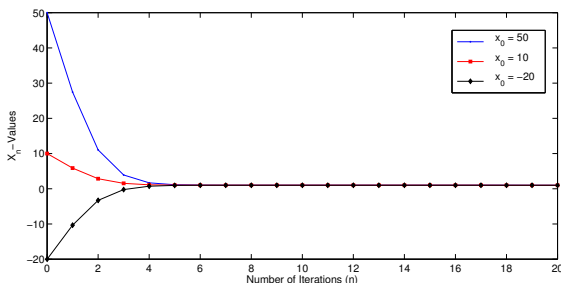


Figure 1. Behavior of x_n for the different initial point $x_0 = 50, 10, -20$.

This shows that for different initial points, each sequence x_n converges to the same solution, i.e., $1 \in F$ as solution of the example.

Next, we would examine the effect of $\alpha_{n,i}$ to rate of convergence and solution by setting three different sequences as follows;

$$\alpha_{n,i} = \begin{cases} \frac{1}{n+1}, & i = 0 \\ \left(1 - \frac{1}{n+1}\right) \frac{1}{2^i}, & i \geq 1 \end{cases},$$

$$\alpha'_{n,i} = \begin{cases} \frac{1}{n+1}, & i = 0 \\ \left(1 - \frac{1}{n+1}\right) \frac{2}{3^i}, & i \geq 1 \end{cases},$$

$$\alpha''_{n,i} = \begin{cases} \frac{1}{3n+1}, & i = 0 \\ \left(1 - \frac{1}{3n+1}\right) \frac{1}{2^i}, & i \geq 1 \end{cases}$$

with the same initial point $x_0 = 30$ and $f(x) = (x + 1)/2$. By Algorithm 1, x_n converges to 1 and the behavior of x_n is showed in Figure 2.

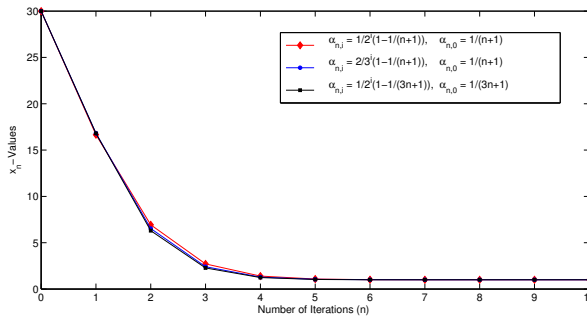


Figure 2. Behavior of x_n for the different sequences $\alpha_{n,i}$, $\alpha'_{n,i}$ and $\alpha''_{n,i}$.

Moreover, Figure 3 shows that by using Algorithm 1, the sequence x_n also converges to 1 for the different contraction functions $f(x) = 0.2(x+1)$, $0.5(x+1)$ and $0.8(x+1)$ with the same initial point $x = -10$, $\alpha_{n,0} = \frac{1}{n+1}$ and $\alpha_{n,i} = \left(1 - \frac{1}{n+1}\right) \frac{1}{2^i}$. The presented results show that $f(x) = 0.5(x+1)$ has the fastest rate of convergence.

The figures 1–3 show that the sequence x_n converge to the same value for the differences of initial points, sequences $\alpha_{n,i}$ and contraction functions. The values of the sequence x_n on the test of the figures 1–3 are shown on Table 1. We note that the differences of initial points and parameter sequences $\alpha_{n,i}$ do not significant indicate the rates of convergence. However, the constants of contraction functions have likely effect on the rates of convergence.

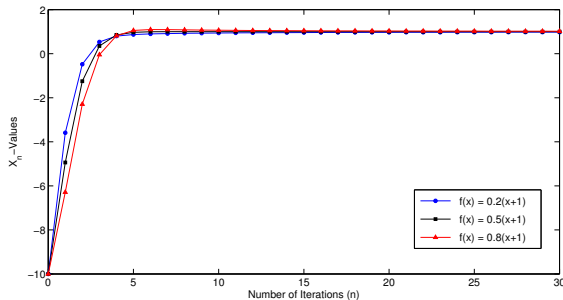


Figure 3. Behavior of x_n for the different contraction functions $f(x) = 0.2(x + 1), 0.5(x + 1)$ and $0.8(x + 1)$.

Table 1. The values of the sequence x_n on the test.

n	$f(x) = (x + 1)/2$			$f(x) = (x + 1)/2$			$\alpha_{n,0} = 1/(n + 1)$		
	$\alpha_{n,0} = \frac{1}{n+1}$			$x_0 = 30$			$\alpha_{n,i} = (1 - \frac{1}{n+1}) \frac{1}{2^i}$		
	$\alpha_{n,i} = (1 - \frac{1}{n+1}) \frac{1}{2^i}$			different α			different $f(x)$		
	different x_0			$\alpha_{n,i}$	$\alpha'_{n,i}$	$\alpha''_{n,i}$	$\frac{2(x+1)}{10}$	$\frac{5(x+1)}{10}$	$\frac{8(x+1)}{10}$
	50	10	-20						
1	28.52	6.055	-10.79	17.29	17.54	18.18	-3.828	-5.178	-6.528
2	11.89	3.001	-3.668	7.447	7.633	7.289	-0.628	-1.445	-2.533
3	4.301	1.606	-0.415	2.953	3.035	2.658	0.479	0.259	-0.185
4	1.809	1.149	0.653	1.479	1.505	1.341	0.784	0.819	0.759
5	1.166	1.031	0.929	1.098	1.105	1.058	0.866	0.963	1.039
6	1.029	1.005	0.987	1.017	1.019	1.008	0.896	0.993	1.094
7	1.005	1.001	0.998	1.003	1.003	1.001	0.913	0.999	1.093
8	1.001	1.00	1.000	1.000	1.000	1.000	0.924	1.000	1.083
9	1.000	1.000	1.000	1.000	1.000	1.000	0.933	1.000	1.073
10	1.000	1.000	1.000	1.000	1.000	1.000	0.940	1.000	1.065
11	1.000	1.000	1.000	1.000	1.000	1.000	0.945	1.000	1.058
12	1.000	1.000	1.000	1.000	1.000	1.000	0.950	1.000	1.053
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
47	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.012
48	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
49	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.003
50	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

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