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Multiple Solutions for a Fractional Laplacian System Involving Critical Sobolev-Hardy Exponents and Homogeneous Term

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Abstract. In this paper, we deal with a class of fractional Laplacian system with critical Sobolev-Hardy exponents and sign-changing weight functions in a bounded domain. By exploiting the Nehari manifold and variational methods, some new existence and multiplicity results are obtain.

Keywords: fractional Laplacian system, Nehari manifold, critical Sobolev-Hardy exponent, homogeneous term.

AMS Subject Classification: 35J20; 35J60.

1 Introduction

In this paper, we mainly study the following system of fractional elliptic equations:

$$\begin{cases} (-\Delta)^s u - \gamma \dfrac{u}{|x|^{2s}} = \lambda f(x) \dfrac{|u|^{q-2}u}{|x|^{\alpha}} + \dfrac{1}{2_s^*(\beta)} \dfrac{F_u(u,v)}{|x|^{\beta}}, & \text{in } \Omega, \\ (-\Delta)^s v - \gamma \dfrac{v}{|x|^{2s}} = \mu g(x) \dfrac{|v|^{q-2}v}{|x|^{\alpha}} + \dfrac{1}{2_s^*(\beta)} \dfrac{F_v(u,v)}{|x|^{\beta}}, & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $s \in (0,1)$, $\Omega \subset \mathbb{R}^N$ $(N > 2s)$ is a smooth bounded domain with $0 \in \Omega$, $2_s^*(\beta) := 2(N-\beta)/(N-2s)$ is the fractional Sobolev-Hardy critical exponent, and the parameters in (1.1) satisfy the following assumptions:

$(\mathcal{H}_0)$ $\quad N > 2s,\ 0 < s < 1,\ 0 \le \alpha, \beta < 2s,\ \lambda, \mu > 0,\ 0 \le \gamma < \gamma_H,\ 1 \le q < 2.$

The operator $(-\Delta)^s$ is the fractional Laplacian which is defined by

$$\begin{aligned}(-\Delta)^s u(x) &= \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy \\ &= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, \quad \forall x \in \mathbb{R}^N.\end{aligned}$$

Notice that the typical feature of the fractional Laplacian operator is nonlocality, that is, the value $(-\Delta)^s u(x)$ at any point $x \in \Omega$ depends not only on the value of u on the Ω, but also on the value of u on the whole $\mathbb{R}^N$, which makes some discussions and calculations difficult. Moreover, the Dirichlet condition in (1.1) is given in $\mathbb{R}^N \setminus \Omega$ and not simply on $\partial\Omega$, which consistently with the nonlocal character of the operator $(-\Delta)^s$, see [1, 3, 16, 17, 19, 21] and the references therein for further details on the fractional Laplacian.

The starting point on the study of the system (1.1) is its scalar version:

$$\begin{cases} (-\Delta)^s u - \gamma \dfrac{u}{|x|^{2s}} = \lambda f(x) \dfrac{|u|^{q-2}u}{|x|^\alpha} + \dfrac{|u|^{2_s^*(\beta)-2}u}{|x|^\beta}, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.2}$$

Concerning the nonlocal problems with critical Sobolev-Hardy exponents, there has been little research up to now, see [18, 20] and the references therein. In particular, Zhang and Hsu [20] concerned the following fractional elliptic system

$$\begin{cases} (-\Delta)^s u - \gamma \dfrac{u}{|x|^{2s}} = \lambda \dfrac{|u|^{q-2}u}{|x|^\alpha} + \dfrac{2\eta}{\eta+\theta} \dfrac{|u|^{\eta-2}u|v|^\theta}{|x|^\beta}, & \text{in } \Omega, \\ (-\Delta)^s v - \gamma \dfrac{v}{|x|^{2s}} = \mu \dfrac{|v|^{q-2}v}{|x|^\alpha} + \dfrac{2\theta}{\eta+\theta} \dfrac{|u|^{\eta}|v|^{\theta-2}v}{|x|^\beta}, & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3}$$

where $1 \le q < 2$, $\eta, \theta > 1$ satisfying $\eta + \theta = 2_s^*(\beta)$. Using the variational method and Nehari manifold method, they found that the problem (1.3) has at least two positive solutions if the parameters λ, $\mu > 0$ satisfied a certain condition. Problems (1.2) and (1.3) aroused the interesting results due to the lack of compactness for involving the critical exponent; hence, the associated energy functionals do not satisfy the Palais-Smale condition in general. Moreover, the explicit formula of the ground states of limiting problem (2.9) is not clear, the standard variational argument is not applicable directly, which is the difficulty for the fractional Laplacian problem with Hardy potential and critical growth.

Motivated by [20], in this paper we focus on the general case f, g possibly change sign in Ω and F positively $2_s^*(\beta)$-homogeneous, we shall complement the results of [18, 20] and extend the results of [11, 12, 13, 14] to the fractional Laplacian operator. Our main tool is the Nehari manifold methods which is similar to the fibering method of Drabek and Pohozaev [5]. We show that the system (1.1) has at least two positive solutions when the parameters λ, μ and weight functions f, g satisfied some certain conditions. It should be mentioned that in [8, 9, 10, 15, 22], some problems involving fractional Laplacian operator were investigated by the Nehari manifold and fibering method.

We look for solutions of (1.1) in the Sobolev space

$$X_0^s(\Omega) = \{u \in H^s(\mathbb{R}^N) : \ u = 0 \ \text{ a.e. in } \ \mathbb{R}^N \backslash \Omega\}$$

with the norm

$$\|u\|_{X_0^s(\Omega)} := \Big(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy \Big)^{\frac{1}{2}}.$$

From (2.2), we employ the following equivalent norm in $X_0^s(\Omega)$:

$$\|u\|_\gamma = \Big(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy - \gamma \int_\Omega \frac{|u|^2}{|x|^{2s}} dx \Big)^{\frac{1}{2}}.$$

Denote by W the product space $W = X_0^s(\Omega) \times X_0^s(\Omega)$ endowed with the norm $\|(u,v)\|_W^2 = \|u\|_\gamma^2 + \|v\|_\gamma^2$. The corresponding energy functional of problem (1.1) is defined on W by

$$I_{\lambda,\mu}(u,v) = \frac{1}{2}\|(u,v)\|_W^2 - \frac{1}{2_s^*(\beta)} \int_\Omega \frac{F(u,v)}{|x|^\beta} dx - \frac{1}{q} Q_{\lambda,\mu}(u,v),$$

where

$$Q_{\lambda,\mu}(u,v) = \lambda \int_\Omega \frac{f(x)|u|^q}{|x|^\alpha} dx + \mu \int_\Omega \frac{g(x)|v|^q}{|x|^\alpha} dx.$$

By standard arguments we can verify $I_{\lambda,\mu} \in C^1(W, \mathbb{R})$. It is well-known that the weak solutions of (1.1) are the critical points of functional $I_{\lambda,\mu}$.

For all $\gamma < \gamma_H$, by $(\mathcal{H}_0)$, the following best Sobolev-type constants are well defined and crucial for the study of (1.1):

$$\Lambda(s,\beta) := \inf_{u \in X_0^s(\Omega)\backslash\{0\}} \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy - \gamma \int_\Omega \frac{|u|^2}{|x|^{2s}} dx}{\Big(\int_\Omega \frac{|u|^{2_s^*(\beta)}}{|x|^\beta} dx \Big)^{\frac{2}{2_s^*(\beta)}}},$$

$$S_F(s,\beta) := \inf_{u \in W\backslash\{(0,0)\}} \|(u,v)\|_W^2 \Big/ \Big(\int_\Omega \frac{F(u,v)}{|x|^\beta} dx \Big)^{\frac{2}{2_s^*(\beta)}}. \tag{1.4}$$

In order to given the relation between $\Lambda(s,\beta)$ and $S_F(s,\beta)$, the following assumptions on F are needed in this paper:

(F_0) $F \in C^1(\mathbb{R}^2, \mathbb{R}^+)$ and $F(tu,tv) = t^{2_s^*(\beta)} F(u,v)$, $\forall t \geq 0$, $(u,v) \in \mathbb{R}^2$;

(F_1) $F(u,0) = F(0,v) = F_u(u,0) = F_v(0,v) = 0$, $\forall u,\ v \in \mathbb{R}$;

(F_2) $F_u(u,v)$, $F_v(u,v)$ are strictly increasing functions for all $(u,v) \in \mathbb{R}^2$.

Then, we have the following result.

Theorem 1. *Suppose* $(\mathcal{H}_0)$ *and* (F_0)*–*(F_2) *hold. Then,* $S_F(s,\beta) = M_F^{-1}\Lambda(s,\beta)$ *and* $S_F(s,\beta)$ *has the minimizers* $(\theta_1 U_\varepsilon(x), \theta_2 U_\varepsilon(x))$*, where* $U_\varepsilon(x)$ *are defined as in* (2.8)*,* θ_1*,* θ_2 *are constants given in* (2.4)*, and* M_F *is defined by*

$$M_F = \max\{F(|u|,|v|)^{\frac{2}{2_s^*(\beta)}} : (u,v) \in \mathbb{R}^2 \text{ and } |u|^2 + |v|^2 = 1\}. \tag{1.5}$$

The other main result of this paper is the following existence and multiple results. To the best of our knowledge, the results are new for the critical fractional Laplacian problem with Hardy potential and homogeneous term.

Set

$$\Lambda_0 = \left(\frac{2-q}{2_s^*(\beta)-q}\right)^{\frac{2}{2_s^*(\beta)-2}} S_F(s,\beta)^{\frac{2_s^*(\beta)}{2_s^*(\beta)-2}} \left(\frac{2_s^*(\beta)-2}{2_s^*(\beta)-q}\right)^{\frac{2}{2-q}} \Lambda(s,\alpha)^{\frac{q}{2-q}}. \tag{1.6}$$

We assume that f, $g : \Omega \to \mathbb{R}$ satisfy

$(\mathcal{H}_1)$ $f, g \in L^p(\Omega, |x|^{-\alpha}dx)$, $f^{\pm} = \max\{\pm f, 0\} \neq 0$ and $g^{\pm} = \max\{\pm g, 0\} \neq 0$, where $p := 2_s^*(\alpha)/(2_s^*(\alpha) - q)$.

$(\mathcal{H}_2)$ For some a_0, $r_0 > 0$ such that $f(x)$, $g(x) \geq a_0$ for $x \in B_{r_0}(0) \subset \Omega$.

Theorem 2. *Suppose* $(\mathcal{H}_0)$*,* $(\mathcal{H}_1)$*–*$(\mathcal{H}_2)$ *and* (F_0)*–*(F_2) *hold. Then,*

(i) If λ*,* $\mu > 0$ *satisfy*

$$0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0,$$

then system (1.1) *has at least one positive solution in* W*.*

(ii) There exists $0 < \Lambda^* < \Lambda_0$ *such that for* λ*,* $\mu > 0$ *and*

$$0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda^*,$$

then problem (1.1) *has at least two positive solutions in* W*.*

Remark 1. There are many homogeneous functions of class C^1, for example: $F(t) = (\sum_{i=1}^k |t_i|^p)^{\frac{2_s^*(\beta)}{p}}$ with $p \geq 1$. If we taking $F(\mathbf{u}) = \frac{2}{\eta+\theta}|u_1|^\eta |u_2|^\theta$ with η, $\theta > 1$, $\eta + \theta = 2_s^*(\beta)$, then Theorems 1.1 and 1.2 in [20] are the special case of our Theorems 1 and 2.

This paper is organized as follows. In Section 2, we introduce the variational setting of the problem and present some norm estimates about the ground states of limiting problem. In Section 3, we investigate the Palais-Smale condition for the energy functional and given the proof of Theorem 1. Some properties about the fibering maps and Nehari manifold are established in Section 4, and Theorem 2 is proved in Sections 5.

Throughout this paper, we will denote by $L^q(\Omega, |x|^\alpha dx)$ the usual weighted $L^q(\Omega)$ space with the weight $|x|^\alpha$ which norm give by $\|\cdot\|_{L^q(\Omega,|x|^\alpha)}$; $O(\varepsilon^t)$ denotes $\frac{|O(\varepsilon^t)|}{\varepsilon^t} \leq C$ and $o(\varepsilon^t)$ means $\frac{|o(\varepsilon^t)|}{\varepsilon^t} \to 0$ as $\varepsilon \to 0$ for $t \geq 0$; $o_n(1)$ means $o_n(1) \to 0$ as $n \to \infty$; The dual space of W will be denoted by W^{-1}; C, C_i will denote various positive constants which may vary from line to line.

2 Preliminaries

First, we give some useful results of fractional Sobolev-Hardy inequality. Assume that $0 \le t \le 2s$. Then, there exists positive constant $C(s,t,N)$ depending on s, t and N, such that

$$C(s,t,N)\left(\int_{\mathbb{R}^N}\frac{|u|^{2_s^*(t)}}{|x|^t}dx\right)^{\frac{2}{2_s^*(t)}} \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy, \tag{2.1}$$

where $2_s^*(t) = \frac{2(N-t)}{N-2s}$. If we set $t = 2s$ in (2.1), we have

$$\gamma_H \int_\Omega \frac{|u|^2}{|x|^{2s}}dx \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy, \quad \forall u \in X_0^s(\Omega), \tag{2.2}$$

where $\gamma_H := 2^{2s}\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}$ is the best fractional Hardy constant.

The following properties about homogeneous function are important and well known:

Lemma 1. *Let $\sigma \ge 1$ and H be a differentiable σ-homogeneous function, then*

(i) for all $(u,v) \in \mathbb{R}^2$, $uH_u(u,v) + vH_v(u,v) = \sigma H(u,v)$;

(ii) there exists $M_H > 0$ such that $|H(u,v)| \le M_H(|u|^\sigma + |v|^\sigma)$ for all $(u,v) \in \mathbb{R}^2$, where

$$M_H = \max\{H(u,v) : (u,v) \in \mathbb{R}^2,\ |u|^\sigma + |v|^\sigma = 1\};$$

(iii) the maximum M_H is attained for some $(u_0,v_0) \in \mathbb{R}^2$, i.e., $|u_0|^\sigma + |v_0|^\sigma = 1$ and $|H(u_0,v_0)| = M_H$;

(iv) $H_u(u,v)$, $H_v(u,v)$ are $(\sigma-1)$ homogenous.

By (F_0) and Lemma 1, we have

$$uF_u(u,v)+vF_v(u,v)=2_s^*(\beta)F(u,v),\ F(u,v) \le \left(M_F(|u|^2+|v|^2)\right)^{\frac{2_s^*(\beta)}{2}}, \tag{2.3}$$

where M_F is given in (1.5). Moreover, from Lemma 1 (iii), there exists $(\theta_1,\theta_2) \in \mathbb{R}^2$ such that

$$\theta_1^2 + \theta_2^2 = 1 \text{ and } M_F = F(\theta_1,\theta_2)^{\frac{2}{2_s^*(\beta)}}. \tag{2.4}$$

Now, we will study $S_F(s,\beta)$ and prove Theorem 1.

Proof of Theorem 1. Let $\{w_n\} \subset X_0^s(\Omega)$ be a minimizing sequence for $\Lambda(s,\beta)$ and (θ_1,θ_2) be defined as in (2.4). Choosing $(u_n,v_n) = (\theta_1 w_n, \theta_2 w_n)$ in (1.4), from (F_0) we have

$$\frac{(\theta_1^2+\theta_2^2)\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w_n(x)-w_n(y)|^2}{|x-y|^{N+2s}}dxdy - \gamma\int_\Omega \frac{|w_n|^2}{|x|^{2s}}dx\right)}{F(\theta_1,\theta_2)^{\frac{2}{2_s^*(\beta)}}\left(\int_\Omega \frac{|w_n|^{2_s^*(\beta)}}{|x|^\beta}dx\right)^{\frac{2}{2_s^*(\beta)}}} \ge S_F(s,\beta). \tag{2.5}$$

Taking $n \to \infty$ in (2.5), we have

$$S_F(s,\beta) \le M_F^{-1} \Lambda(s,\beta). \tag{2.6}$$

On the other hand, let $\{(u_n, v_n)\} \subset W \backslash \{(0,0)\}$ be a minimizing sequence for $S_F(s,\beta)$, from Proposition 1 in [4], we have

$$\begin{aligned}\int_\Omega \frac{F(u_n,v_n)}{|x|^\beta} dx &= \int_\Omega F\Big(\frac{u_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}, \frac{v_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big) dx \\ &\le F\Big(\Big\|\frac{u_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big\|_{L^{2_s^*(\beta)}}, \Big\|\frac{v_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big\|_{L^{2_s^*(\beta)}}\Big).\end{aligned}$$

Set

$$\mathcal{A} = 1/\Big(\Big\|\frac{u_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big\|^2_{L^{2_s^*(\beta)}} + \Big\|\frac{v_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big\|^2_{L^{2_s^*(\beta)}}\Big)^{1/2}.$$

We have

$$\Big\|\frac{\mathcal{A}u_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big\|^2_{L^{2_s^*(\beta)}} + \Big\|\frac{\mathcal{A}v_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big\|^2_{L^{2_s^*(\beta)}} = 1.$$

Then,

$$\begin{aligned}&\frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\big(\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}} + \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}\big)dxdy - \gamma\int_\Omega \frac{|u_n|^2+|v_n|^2}{|x|^{2s}}dx}{\big(\int_\Omega \frac{F(u_n,v_n)}{|x|^\beta}dx\big)^{\frac{2}{2_s^*(\beta)}}} \\
&\ge \frac{\Lambda(s,\beta)\big(\int_\Omega \frac{|u_n|^{2_s^*(\beta)}}{|x|^\beta}dx\big)^{\frac{2}{2_s^*(\beta)}} + \Lambda(s,\beta)\big(\int_\Omega \frac{|v_n|^{2_s^*(\beta)}}{|x|^\beta}dx\big)^{\frac{2}{2_s^*(\beta)}}}{\Big(F\Big(\Big\|\frac{u_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big\|_{L^{2_s^*(\beta)}}, \Big\|\frac{v_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big\|_{L^{2_s^*(\beta)}}\Big)\Big)^{\frac{2}{2_s^*(\beta)}}} \\
&= \Lambda(s,\beta)\frac{\Big\|\frac{u_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big\|^2_{L^{2_s^*(\beta)}} + \Big\|\frac{v_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big\|^2_{L^{2_s^*(\beta)}}}{\Big(F\Big(\Big\|\frac{u_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big\|_{L^{2_s^*(\beta)}}, \Big\|\frac{v_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big\|_{L^{2_s^*(\beta)}}\Big)\Big)^{\frac{2}{2_s^*(\beta)}}} \\
&= \Lambda(s,\beta)\frac{\Big\|\frac{\mathcal{A}u_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big\|^2_{L^{2_s^*(\beta)}} + \Big\|\frac{\mathcal{A}v_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big\|^2_{L^{2_s^*(\beta)}}}{\Big(F\Big(\Big\|\frac{\mathcal{A}u_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big\|_{L^{2_s^*(\beta)}}, \Big\|\frac{\mathcal{A}v_n}{|x|^{\frac{\beta}{2_s^*(\beta)}}}\Big\|_{L^{2_s^*(\beta)}}\Big)\Big)^{\frac{2}{2_s^*(\beta)}}} \\
&\ge M_F^{-1}\Lambda(s,\beta).\end{aligned}$$

Passing to the limit in the above inequality, we have

$$S_F(s,\beta) \ge M_F^{-1}\Lambda(s,\beta). \tag{2.7}$$

Hence, (2.6) and (2.7) given the proof of Theorem 1. □

For the best constant $\Lambda(s,\beta)$, from [6, 7], we know that, for all $0<s<1$ and $0\le\gamma<\gamma_H$, the best constant $\Lambda(s,\beta)$ is achieved by the form

$$U_\varepsilon(x)=\varepsilon^{-\frac{N-2s}{2}}U(x/\varepsilon), \qquad \forall\varepsilon>0, \tag{2.8}$$

where $U\in C^1(\mathbb{R}^N\backslash\{0\})$ is a positive, radially symmetric, radially decreasing ground state solution of the limit problem:

$$\begin{cases}(-\Delta)^s u-\gamma\dfrac{u}{|x|^{2s}}=\dfrac{u^{2_s^*(\beta)-1}}{|x|^\beta}, & \text{in } \mathbb{R}^N,\\ u\ge 0, \quad u\not\equiv 0, & \text{in } \mathbb{R}^N.\end{cases} \tag{2.9}$$

Moreover, at zero and infinity, the solution U satisfies

$$\lim_{|x|\to 0}|x|^{a(\gamma)}U(x)=\lambda_0>0 \quad \text{and} \quad \lim_{|x|\to\infty}|x|^{b(\gamma)}U(x)=\lambda_\infty>0,$$

where $a(\gamma)$ and $b(\gamma)$ are solutions of the equation

$$\Psi_{N,s}(\beta)=2^{2s}\frac{\Gamma(\frac{N-\beta}{2})\Gamma(\frac{2s+\beta}{2})}{\Gamma(\frac{N-2s-\beta}{2})\Gamma(\frac{\beta}{2})}-\gamma=0$$

and satisfy $0\le a(\gamma)<\frac{N-2s}{2}<b(\gamma)\le N-2s$. By a direct computation, we get

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|U_\varepsilon(x)-U_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy-\gamma\int\limits_{\mathbb{R}^N}\frac{|U_\varepsilon|^2}{|x|^{2s}}dx=\int\limits_{\mathbb{R}^N}\frac{|U_\varepsilon|^{2_s^*(\beta)}}{|x|^\beta}dx=\Lambda(s,\beta)^{\frac{2_s^*(\beta)}{2_s^*(\beta)-2}}.$$

Take $\rho\in(0,r_0)$ small enough such that $B_\rho(0)\subset\Omega$, $B_\rho(0)=\{x\in\mathbb{R}^N: |x|<\rho\}$, where r_0 be given in $(\mathcal{H}_2)$. Choose the radial cut-off function $\varphi\in C_0^\infty(\Omega)$ such that $0\le\varphi(x)\le 1$ if $B_\rho(0)$, $\varphi(x)=1$ in $B_{\frac{\rho}{2}}(0)$ and $\varphi(x)=0$ if $B_\rho(0)^c$. Set

$$u_\varepsilon(x)=\varphi(x)U_\varepsilon(x), \quad \forall\varepsilon>0. \tag{2.10}$$

Proposition 1. *(See [20], Proposition 2.3)Assume that $0<s<1$, $0\le\gamma<\gamma_H$, $0\le\alpha$, $\beta<2s$ and $1\le q<2_s^*(\alpha)$. Then, the following estimates hold as $\varepsilon\to 0^+$:*

$$\|u_\varepsilon\|_\gamma^2=\Lambda(s,\beta)^{\frac{N-\beta}{2s-\beta}}+O(\varepsilon^{2b(\gamma)+2s-N}); \tag{2.11}$$

$$\int_\Omega\frac{|u_\varepsilon|^{2_s^*(\beta)}}{|x|^\beta}dx=\Lambda(s,\beta)^{\frac{N-\beta}{2s-\beta}}+O(\varepsilon^{2_s^*(\beta)b(\gamma)+\beta-N}); \tag{2.12}$$

$$\int_\Omega\frac{|u_\varepsilon|^q}{|x|^\alpha}dx=\begin{cases}C\varepsilon^{N-\alpha-\frac{q(N-2s)}{2}}, & \text{if } q>(N-\alpha)/b(\gamma),\\ C\varepsilon^{N-\alpha-\frac{q(N-2s)}{2}}|\ln\varepsilon|, & \text{if } q=(N-\alpha)/b(\gamma),\\ C\varepsilon^{q(b(\gamma)-\frac{N-2s}{2}}, & \text{if } q<(N-\alpha)/b(\gamma).\end{cases} \tag{2.13}$$

3 The Palais-Smale condition

In this section, we show that the functional $I_{\lambda,\mu}$ satisfies $(PS)_c$ conditions.

Definition 1. Let $c \in \mathbb{R}$, W be a Banach space and $I_{\lambda,\mu} \in C^1(W,\mathbb{R})$. Then $\{(u_n, v_n)\}$ is a $(PS)_c$ sequence in W for $I_{\lambda,\mu}$ if $I_{\lambda,\mu}(u_n, v_n) = c + o_n(1)$ and $I'_{\lambda,\mu}(u_n, v_n) = o_n(1)$ strongly in W^{-1} as $n \to \infty$. We say that $I_{\lambda,\mu}$ satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence $\{(u_n, v_n)\}$ for $I_{\lambda,\mu}$ admits a convergent subsequence.

By the Hölder and Sobolev-Hardy inequalities, for all $u \in X_0^s(\Omega)$, we get

$$\begin{aligned}\int_\Omega \frac{f(x)|u|^q}{|x|^\alpha}dx &= \int_\Omega \frac{|u|^q}{|x|^{\frac{q}{2_s^*(\alpha)}\alpha}} \cdot \frac{f(x)}{|x|^{(1-\frac{q}{2_s^*(\alpha)})\alpha}}dx \\ &\le \Big(\int_\Omega \frac{|f|^{\frac{2_s^*(\alpha)}{2_s^*(\alpha)-q}}}{|x|^\alpha}dx\Big)^{\frac{2_s^*(\alpha)-q}{2_s^*(\alpha)}} \Big(\int_\Omega \frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}dx\Big)^{\frac{q}{2_s^*(\alpha)}} \\ &= \|f\|_{L^{\frac{2_s^*(\alpha)}{2_s^*(\alpha)-q}}(\Omega,|x|^{-\alpha})} \Lambda(s,\alpha)^{-\frac{q}{2}}\|u\|_\gamma^q.\end{aligned} \tag{3.1}$$

Then,

$$\begin{aligned}Q_{\lambda,\mu}(u,v) &\le \Lambda(s,\alpha)^{-\frac{q}{2}}(\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})}\|u\|_\gamma^q + \mu\|g\|_{L^p(\Omega,|x|^{-\alpha})}\|v\|_\gamma^q) \\ &= \Big(\Big[\frac{2}{q}\Big(\frac12 - \frac{1}{2_s^*(\beta)}\Big)\Big(\frac1q - \frac{1}{2_s^*(\beta)}\Big)^{-1}\Big]^{\frac{q}{2}}\|u\|_\gamma^q\Big) \\ &\quad\times\Big(\Big[\frac{2}{q}\Big(\frac12 - \frac{1}{2_s^*(\beta)}\Big)\Big(\frac1q - \frac{1}{2_s^*(\beta)}\Big)^{-1}\Big]^{-\frac{q}{2}}\Lambda(s,\alpha)^{-\frac{q}{2}}\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})}\Big) \\ &\quad+\Big(\Big[\frac{2}{q}\Big(\frac12 - \frac{1}{2_s^*(\beta)}\Big)\Big(\frac1q - \frac{1}{2_s^*(\beta)}\Big)^{-1}\Big]^{\frac{q}{2}}\|v\|_\gamma^q\Big) \\ &\quad\times\Big(\Big[\frac{2}{q}\Big(\frac12 - \frac{1}{2_s^*(\beta)}\Big)\Big(\frac1q - \frac{1}{2_s^*(\beta)}\Big)^{-1}\Big]^{-\frac{q}{2}}\Lambda(s,\alpha)^{-\frac{q}{2}}\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})}\Big) \\ &\le \Big(\frac12 - \frac{1}{2_s^*(\beta)}\Big)\Big(\frac1q - \frac{1}{2_s^*(\beta)}\Big)^{-1}\|(u,v)\|_W^2 \\ &\quad+C_*\Big((\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}\Big),\end{aligned} \tag{3.2}$$

where

$$C_* = \frac{2-q}{2}\Big(\frac{2_s^*(\beta)-q}{2_s^*(\beta)-2}\Big)^{\frac{q}{2-q}}\Lambda(s,\alpha)^{-\frac{q}{2-q}} > 0.$$

Lemma 2. *If $\{(u_n, v_n)\}$ is a $(PS)_c$ sequence for $I_{\lambda,\mu}$ with $(u_n, v_n) \rightharpoonup (u,v)$ weakly in W. Then, $I'_{\lambda,\mu}(u,v) = 0$ and there exists a positive constant C_0 depending on q, s, α, β and N such that*

$$I_{\lambda,\mu}(u,v) \ge -C_0\Big((\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}\Big).$$

Proof. Since $\{(u_n, v_n)\}$ is a $(PS)_c$ sequence for $I_{\lambda,\mu}$ with $(u_n, v_n) \rightharpoonup (u,v)$ in W, it is easy to check that $I'_{\lambda,\mu}(u,v) = 0$. In particular, we get that

$$\langle I'_{\lambda,\mu}(u,v), (u,v)\rangle = 0,$$

namely

$$\|(u,v)\|_W^2 = \int_\Omega \frac{F(u,v)}{|x|^\beta}dx + Q_{\lambda,\mu}(u,v).$$

Then, from (3.2), we have

$$\begin{aligned} I_{\lambda,\mu}(u,v) &= \Big(\frac{1}{2} - \frac{1}{2_s^*(\beta)}\Big)\|(u,v)\|_W^2 - \Big(\frac{1}{q} - \frac{1}{2_s^*(\beta)}\Big)Q_{\lambda,\mu}(u,v) \\ &\geq -\Big(\frac{1}{q} - \frac{1}{2_s^*(\beta)}\Big)C_*\Big((\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}\Big) \\ &= -C_0\Big((\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}\Big), \end{aligned}$$

where $C_0 = \frac{(2_s^*(\beta)-q)(2-q)}{2q2_s^*(\beta)}\Big(\frac{2_s^*(\beta)-q}{2_s^*(\beta)-2}\Big)^{\frac{q}{2-q}}\Lambda(s,\alpha)^{-\frac{q}{2-q}}$ is a positive constant depending on q, s, α, β and N. $\square$

Lemma 3. *$I_{\lambda,\mu}$ satisfies the $(PS)_c$ condition for all $c < c_\infty$, where*

$$\begin{aligned} c_\infty :=& \frac{2s-\beta}{2(N-\beta)}S_F(s,\beta)^{\frac{2_s^*(\beta)}{2_s^*(\beta)-2}} \\ &- C_0\Big((\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}\Big). \end{aligned} \tag{3.3}$$

Proof. Let $\{(u_n,v_n)\}$ be a $(PS)_c$ sequence for $c \in (-\infty, c_\infty)$. Similarly to the proof of [13, Lemma 2.3], it is easy to see that $\{(u_n,v_n)\}$ is bounded in W. Then, there exist a subsequence still denoted by $\{(u_n,v_n)\}$ and $(u,v) \in W$ such that $(u_n,v_n) \rightharpoonup (u,v)$ weakly in W, and

$$\begin{cases} u_n \rightharpoonup u, \quad v_n \rightharpoonup v \ \text{ weakly in } \ L^{2_s^*(\beta)}(\Omega,|x|^{-\beta}dx), \\ u_n \to u, \quad v_n \to v \ \text{ strongly in } \ L^q(\Omega,|x|^{-\alpha}dx), \ \forall 1 \leq q < 2_s^*(\alpha), \\ u_n(x) \to u(x), \quad v_n(x) \to v(x) \ \text{ a.e. in } \ \Omega. \end{cases} \tag{3.4}$$

Hence, from (3.4), it is easy to verify that $I'_{\lambda,\mu}(u,v) = 0$ and

$$Q_{\lambda,\mu}(u_n,v_n) = Q_{\lambda,\mu}(u,v) + o_n(1) \quad (n \to \infty). \tag{3.5}$$

Set $\tilde{u}_n = u_n - u$, $\tilde{v}_n = v_n - v$. By Brézis-Lieb lemma [19], we get

$$\|(u_n,v_n)\|_W^2 = \|(u,v)\|_W^2 + \|(\tilde{u}_n,\tilde{v}_n)\|_W^2 + o_n(1). \tag{3.6}$$

By the same methods as in [4, Lemma 8], we obtain

$$\int_\Omega \frac{F(u_n,v_n)}{|x|^\beta}dx = \int_\Omega \frac{F(u,v)}{|x|^\beta}dx + \int_\Omega \frac{F(\tilde{u}_n,\tilde{v}_n)}{|x|^\beta}dx + o_n(1). \tag{3.7}$$

Using (3.5), (3.6) and (3.7), we have

$$c = \frac{1}{2}\|(\tilde{u}_n,\tilde{v}_n)\|_W^2 - \frac{1}{2_s^*(\beta)}\int_\Omega \frac{F(\tilde{u}_n,\tilde{v}_n)}{|x|^\beta}dx + I_{\lambda,\mu}(u,v) + o_n(1) \tag{3.8}$$

and

$$o_n(1) = \|(\tilde{u}_n, \tilde{v}_n)\|_W^2 - \int_\Omega \frac{F(\tilde{u}_n, \tilde{v}_n)}{|x|^\beta} dx. \tag{3.9}$$

Thus, we may assume that

$$\int_\Omega \frac{F(\tilde{u}_n, \tilde{v}_n)}{|x|^\beta} dx \to l, \quad \|(\tilde{u}_n, \tilde{v}_n)\|_W^2 \to l \geq 0 \quad \text{as } n \to \infty.$$

If $l = 0$, the proof is completed. Assume that $l > 0$, then from (3.9) we have

$$S_F(s,\beta) l^{\frac{2}{2_s^*(\beta)}} = S_F(s,\beta) \Big(\lim_{n\to\infty} \int_\Omega \frac{F(\tilde{u}_n, \tilde{v}_n)}{|x|^\beta} dx \Big)^{\frac{2}{2_s^*(\beta)}} \leq \lim_{n\to\infty} \|(\tilde{u}_n, \tilde{v}_n)\|_W^2 = l,$$

which implies that $l \geq S_F(s,\beta)^{\frac{2_s^*(\beta)}{2_s^*(\beta)-2}}$. Hence, from (3.8) and Lemma 2, we have

$$\begin{aligned} c = I_{\lambda,\mu}(u_n, v_n) + o_n(1) = \Big(\frac{1}{2} - \frac{1}{2_s^*(\beta)} \Big) l + I_{\lambda,\mu}(u,v) + o_n(1) \geq \frac{2s-\beta}{2(N-\beta)} \\ \times S_F(s,\beta)^{\frac{2_s^*(\beta)}{2_s^*(\beta)-2}} - C_0 \Big((\lambda \|f\|_{L^p(\Omega, |x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} \Big), \end{aligned}$$

which contradicts $c < c_\infty$. The proof is completed. □

4 Nehari manifold

Since $I_{\lambda,\mu}$ is not bounded below on W, we need to study $I_{\lambda,\mu}$ on the Nehari manifold

$$\mathcal{N}_{\lambda,\mu} = \{(u,v) \in W \backslash \{(0,0)\} \, : \, \langle I'_{\lambda,\mu}(u,v) \, , \, (u,v) \rangle = 0\}.$$

Note that $\mathcal{N}_{\lambda,\mu}$ contains all nonzero solution of (1.1), and $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if

$$\|(u,v)\|_W^2 - \int_\Omega \frac{F(u,v)}{|x|^\beta} dx - Q_{\lambda,\mu}(u,v) = 0. \tag{4.1}$$

Lemma 4. *$I_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. Let $(u,v) \in \mathcal{N}_{\lambda,\mu}$, by (3.1), the Sobolev inequality and Hölder inequality, we find

$$Q_{\lambda,\mu}(u,v) \leq C_1 \|(u,v)\|_W^q, \tag{4.2}$$

where

$$C_1 = \Big[(\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} \Big]^{\frac{2-q}{2}} \Lambda(s,\alpha)^{-\frac{q}{2}} > 0.$$

From (4.1) and (4.2), we get

$$\begin{aligned} I_{\lambda,\mu}(u,v) &= \Big(\frac{1}{2} - \frac{1}{2_s^*(\beta)} \Big) \|(u,v)\|_W^2 - \Big(\frac{1}{q} - \frac{1}{2_s^*(\beta)} \Big) Q_{\lambda,\mu}(u,v) \\ &\geq \frac{2_s^*(\beta) - 2}{2\, 2_s^*(\beta)} \|(u,v)\|_W^2 - \frac{2_s^*(\beta) - q}{q\, 2_s^*(\beta)} C_1 \|(u,v)\|_W^q. \end{aligned}$$

As $1 \le q < 2$, $I_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$. $\square$

Define $\Psi_{\lambda,\mu}(u,v) := \langle I'_{\lambda,\mu}(u,v)\,,\,(u,v)\rangle$, then for all $(u,v) \in \mathcal{N}_{\lambda,\mu}$, we get

$$\begin{aligned}\langle \Psi'_{\lambda,\mu}(u,v)\,,\,(u,v)\rangle &= 2\|(u,v)\|_W^2 - qQ_{\lambda,\mu}(u,v) - 2_s^*(\beta)\int_\Omega \frac{F(u,v)}{|x|^\beta}dx \\ &= (2-q)\|(u,v)\|_W^2 - (2_s^*(\beta)-q)\int_\Omega \frac{F(u,v)}{|x|^\beta}dx \qquad (4.3)\\ &= (2_s^*(\beta)-q)Q_{\lambda,\mu}(u,v) - (2_s^*(\beta)-2)\|(u,v)\|_W^2. \qquad (4.4)\end{aligned}$$

We split $\mathcal{N}_{\lambda,\mu}$ into three parts:

$$\begin{aligned}\mathcal{N}^+_{\lambda,\mu} &= \{(u,v) \in \mathcal{N}_{\lambda,\mu} \,:\, \langle \Psi'_{\lambda,\mu}(u,v)\,,\,(u,v)\rangle > 0\},\\ \mathcal{N}^0_{\lambda,\mu} &= \{(u,v) \in \mathcal{N}_{\lambda,\mu} \,:\, \langle \Psi'_{\lambda,\mu}(u,v)\,,\,(u,v)\rangle = 0\},\\ \mathcal{N}^-_{\lambda,\mu} &= \{(u,v) \in \mathcal{N}_{\lambda,\mu} \,:\, \langle \Psi'_{\lambda,\mu}(u,v)\,,\,(u,v)\rangle < 0\}.\end{aligned}$$

We now present some important properties of $\mathcal{N}_{\lambda,\mu}$, $\mathcal{N}^+_{\lambda,\mu}$, $\mathcal{N}^-_{\lambda,\mu}$.

Lemma 5. *Assume that (u_0,v_0) is a local minimizer for $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ and $(u_0,v_0) \notin \mathcal{N}^0_{\lambda,\mu}$. Then $I'_{\lambda,\mu}(u_0,v_0) = 0$ in W^{-1}.*

Proof. The proof is similar to that of [20, Lemma 3.4] and the details are omitted. $\square$

Lemma 6. *$\mathcal{N}^0_{\lambda,\mu} = \emptyset$ for all λ, $\mu > 0$ satisfy $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$, where Λ_0 is the same as that in* (1.6).

Proof. We argue by contradiction. Assume that there exist λ and $\mu > 0$ with $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$ such that $\mathcal{N}^0_{\lambda,\mu} \neq \emptyset$. Then, by (4.3) and (4.4), for $(u,v) \in \mathcal{N}^0_{\lambda,\mu}$, we have

$$\|(u,v)\|_W^2 = \frac{2_s^*(\beta)-q}{2-q}\int_\Omega \frac{F(u,v)}{|x|^\beta}dx,\ \ \|(u,v)\|_W^2 = \frac{2_s^*(\beta)-q}{2_s^*(\beta)-2}Q_{\lambda,\mu}(u,v). \qquad (4.5)$$

According to (2.3) and the Minkowshi inequality, we obtain that

$$\begin{aligned}\int_\Omega \frac{F(u,v)}{|x|^\beta}dx &\le M_F^{\frac{2_s^*(\beta)}{2}}\int_\Omega \frac{(|u|^2+|v|^2)^{\frac{2_s^*(\beta)}{2}}}{|x|^\beta}dx\\ &\le M_F^{\frac{2_s^*(\beta)}{2}}\Big[\Big(\int_\Omega \frac{|u|^{2_s^*(\beta)}}{|x|^\beta}dx\Big)^{\frac{2}{2_s^*(\beta)}} + \Big(\int_\Omega \frac{|v|^{2_s^*(\beta)}}{|x|^\beta}dx\Big)^{\frac{2}{2_s^*(\beta)}}\Big]^{\frac{2_s^*(\beta)}{2}}\\ &\le M_F^{\frac{2_s^*(\beta)}{2}}\Big(\Lambda(s,\beta)^{-1}\|u\|_\gamma^2 + \Lambda(s,\beta)^{-1}\|v\|_\gamma^2\Big)^{\frac{2_s^*(\beta)}{2}}\\ &= \Big(\frac{\Lambda(s,\beta)}{M_F}\Big)^{-\frac{2_s^*(\beta)}{2}}\Big(\|u\|_\gamma^2 + \|v\|_\gamma^2\Big)^{\frac{2_s^*(\beta)}{2}} = S_F(s,\beta)^{-\frac{2_s^*(\beta)}{2}}\|(u,v)\|_W^{2_s^*(\beta)}, \qquad (4.6)\end{aligned}$$

which and (4.5) leads that

$$\|(u,v)\|_W \geq \Big(\frac{2-q}{2_s^*(\beta)-q} S_F(s,\beta)^{\frac{2_s^*(\beta)}{2}}\Big)^{\frac{1}{2_s^*(\beta)-2}}. \tag{4.7}$$

On the other hand, by (4.2) and (4.5), we find that

$$\begin{aligned}\|(u,v)\|_W \leq & \Big(\frac{2_s^*(\beta)-q}{2_s^*(\beta)-2}\Big)^{\frac{1}{2-q}} \Lambda(s,\alpha)^{-\frac{q}{2(2-q)}} \\ & \times \Big((\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}\Big)^{\frac{1}{2}}.\end{aligned} \tag{4.8}$$

Consequently, (4.7) and (4.8) implies

$$(\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} \geq \Lambda_0,$$

which contradicts to $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$. This completes the proof of Lemma 6. □

By Lemmas 4 and 6, for each λ, $\mu > 0$ with $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$, we can write $\mathcal{N}_{\lambda,\mu} = \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}$. Define

$$c_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(u,v),\ c^+_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}} I_{\lambda,\mu}(u,v),\ c^-_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}} I_{\lambda,\mu}(u,v).$$

Then, we have the following results.

Lemma 7. *The following results hold.*

(1) If $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$, *then* $c_{\lambda,\mu} \leq c^+_{\lambda,\mu} < 0$.

(2) There exists

$$\Lambda_1 := \Big(\frac{q(2_s^*(\beta)-2)}{2(2_s^*(\beta)-q)}\Big)^{\frac{2}{2-q}} \Lambda(s,\alpha)^{\frac{q}{2-q}} S_F(s,\beta)^{\frac{2_s^*(\beta)}{2_s^*(\beta)-2}} \Big(\frac{2-q}{2_s^*(\beta)-q}\Big)^{\frac{2}{2_s^*(\beta)-2}}$$

such that for all λ, $\mu > 0$ *with*

$$0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_1,$$

then $c^-_{\lambda,\mu} \geq c_0$ *for some* $c_0 > 0$.

Proof. (1) Suppose $(u,v) \in \mathcal{N}^+_{\lambda,\mu}$. By (4.3), we get

$$(2-q)\|(u,v)\|^2_W > (2_s^*(\beta)-q)\int_\Omega \frac{F(u,v)}{|x|^\beta}dx. \tag{4.9}$$

According to (4.1) and (4.9), we get

$$\begin{aligned} I_{\lambda,\mu}(u,v) &= \Big(\frac{1}{2}-\frac{1}{q}\Big)\|(u,v)\|_W^2 - \Big(\frac{1}{2_s^*(\beta)}-\frac{1}{q}\Big)\int_\Omega \frac{F(u,v)}{|x|^\beta}dx \\ &\le \Big[\Big(\frac{1}{2}-\frac{1}{q}\Big)+\Big(\frac{1}{q}-\frac{1}{2_s^*(\beta)}\Big)\frac{2-q}{2_s^*(\beta)-q}\Big]\|(u,v)\|_W^2 \\ &= -\frac{(2-q)(2_s^*(\beta)-2)}{2q2_s^*(\beta)}\|(u,v)\|_W^2 < 0. \end{aligned}$$

Then, by the definition of $c_{\lambda,\mu}$, $c^+_{\lambda,\mu}$, we can deduce that $c_{\lambda,\mu} \le c^+_{\lambda,\mu} < 0$.

(2) Suppose $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$. From (4.3), it follows that

$$(2-q)\|(u,v)\|_W^2 < (2_s^*(\beta)-q)\int_\Omega \frac{F(u,v)}{|x|^\beta}dx.$$

This and (4.6) yield

$$\|(u,v)\|_W > \Big(\frac{2-q}{2_s^*(\beta)-q}\Big)^{\frac{1}{2_s^*(\beta)-2}} S_F(s,\beta)^{\frac{2_s^*(\beta)}{2(2_s^*(\beta)-2)}}. \tag{4.10}$$

By the proof of Lemma 4 and (4.10), we infer that

$$\begin{aligned} &I_{\lambda,\mu}(u,v) \\ &\ge \|(u,v)\|_W^q\Big[\Big(\frac{1}{2}-\frac{1}{2_s^*(\beta)}\Big)\|(u,v)\|_W^{2-q} - \Big(\frac{2_s^*(\beta)-q}{q2_s^*(\beta)}\Big) \\ &\quad\times\Big((\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}\Big)^{\frac{2-q}{2}}\Lambda(s,\alpha)^{-\frac{q}{2}}\Big] \\ &> \|(u,v)\|_W^q\Big[\frac{2_s^*(\beta)-2}{22_s^*(\beta)}\Big(\frac{2-q}{2_s^*(\beta)-q}\Big)^{\frac{2-q}{2_s^*(\beta)-2}} S_F(s,\beta)^{\frac{2_s^*(\beta)(2-q)}{2(2_s^*(\beta)-2)}} - \Big(\frac{2_s^*(\beta)-q}{q2_s^*(\beta)}\Big) \\ &\quad\times\Big((\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}\Big)^{\frac{2-q}{2}}\Lambda(s,\alpha)^{-\frac{q}{2}}\Big]. \end{aligned}$$

Then, if $(\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_1$, we get $I_{\lambda,\mu}(u,v) \ge c_0$ for all $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$, where $c_0 = c(q,s,\alpha,\beta,N)$ is a positive constant. □

For $t > 0$, we define the fibering maps $\Phi_{u,v}(t) = I_{\lambda,\mu}(tu,tv)$. Then,

$$\Phi'_{u,v}(t) = t\|(u,v)\|_W^2 - t^{2_s^*(\beta)-1}\int_\Omega \frac{F(u,v)}{|x|^\beta}dx - t^{q-1}Q_{\lambda,\mu}(u,v).$$

For $(u,v) \in \mathcal{N}_{\lambda,\mu}$, we get $\Phi'_{u,v}(1) = \langle I'_{\lambda,\mu}(u,v), (u,v)\rangle$, which implies that $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if $\Phi'_{u,v}(1) = 0$, and more generally $(tu,tv) \in \mathcal{N}_{\lambda,\mu}$ if and only if $\Phi'_{u,v}(t) = 0$. That is, the elements in $\mathcal{N}_{\lambda,\mu}$ correspond to stationary points of the fibering maps $\Phi_{u,v}(t)$.

For each $(u,v) \in W$ with $\int_\Omega \frac{F(u,v)}{|x|^\beta}dx > 0$, set

$$t_{\max} = \left(\frac{(2-q)\|(u,v)\|_W^2}{(2_s^*(\beta)-q)\int_\Omega \frac{F(u,v)}{|x|^\beta}dx}\right)^{\frac{1}{2_s^*(\beta)-2}} > 0.$$

Then, the following lemma holds.

Lemma 8. *If* $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$, *then for every* $(u,v) \in W\backslash\{(0,0)\}$ *with* $\int_\Omega \frac{F(u,v)}{|x|^\beta}dx > 0$ *and* $Q_{\lambda,\mu}(u,v) > 0$, *there exist unique* t_1 *and* $t_2 > 0$ *such that* $(t_1u, t_1v) \in \mathcal{N}^+_{\lambda,\mu}$, $(t_2u,t_2v) \in \mathcal{N}^-_{\lambda,\mu}$. *Moreover, we have* $0 < t_1 < t_{max} < t_2$, $I_{\lambda,\mu}(t_1u,t_1v) = \inf_{t\in[0,t_{\max}]} I_{\lambda,\mu}(tu,tv)$ *and* $I_{\lambda,\mu}(t_2u,t_2v) = \sup_{t\in[0,\infty)} I_{\lambda,\mu}(tu,tv)$.

Proof. The proof is similar to [20, Lemma 3.2] and the details are omitted. □

5 Proof of Theorem 2

Before giving the proof of Theorem 2, we need the following lemma.

Lemma 9. *Suppose* $(\mathcal{H}_0)$–$(\mathcal{H}_2)$ *and* $(\mathcal{F}_0)$–$(\mathcal{F}_2)$ *hold. The following facts hold.*

(i) If $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$, *then there exists a* $(PS)_{c_{\lambda,\mu}}$ *-sequence* $\{(u_n,v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ *for* $I_{\lambda,\mu}$.

(ii) If $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_1$, *then there exists a* $(PS)_{c^-_{\lambda,\mu}}$ *-sequence* $\{(u_n,v_v)\} \subset \mathcal{N}^-_{\lambda,\mu}$ *for* $I_{\lambda,\mu}$.

Proof. The proof is almost the same as in [18, Proposition 3.8] and the details are omitted. □

Now, we establish the existence of a local minimizer for $I_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}$.

Theorem 3. *Assume that* $(\mathcal{H}_0)$–$(\mathcal{H}_2)$ *and* (F_0)–(F_2). *If*

$$0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0,$$

then $I_{\lambda,\mu}$ *has a minimizer* $(u_1,v_1) \in \mathcal{N}^+_{\lambda,\mu}$ *such that* (u_1,v_1) *is a positive solution of* (1.1) *and* $I_{\lambda,\mu}(u_1,v_1) = c_{\lambda,\mu} = c^+_{\lambda,\mu} < 0$.

Proof. By Lemma 9 (i), there exists a minimizing sequence $\{(u_n,v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(u_n,v_n) = c_{\lambda,\mu} + o_n(1) \quad \text{and} \quad I'_{\lambda,\mu}(u_n,v_n) = o_n(1). \tag{5.1}$$

Since $I_{\lambda,\mu}$ is coercive on $\mathcal{N}_{\lambda,\mu}$, we get that $\{(u_n,v_n)\}$ is bounded in W. Passing to a subsequence, still denoted by $\{(u_n,v_n)\}$, we can assume that there exists $(u_1,v_1) \in W$ such that $(u_n,v_n) \rightharpoonup (u_1,v_1)$ weakly in W and

$$\begin{cases} u_n \rightharpoonup u_1, \quad v_n \rightharpoonup v_1 \text{ weakly in } L^{2^*_s(\beta)}(\Omega,|x|^{-\beta}dx), \\ u_n \to u_1, \quad v_n \to v_1 \text{ strongly in } L^q(\Omega,|x|^{-\alpha}dx), \ \forall\, 1 \le q < 2^*_\alpha(s), \\ u_n(x) \to u_1(x), \quad v_n(x) \to v_1(x) \text{ a.e. in } \Omega. \end{cases} \tag{5.2}$$

This implies that

$$Q_{\lambda,\mu}(u_n,v_n) = Q_{\lambda,\mu}(u_1,v_1) + o_n(1). \tag{5.3}$$

First, we claim that (u_1, v_1) is a nontrivial weak solution of (1.1). From (5.1), (5.2) and (5.3), it is easy to verify that (u_1, v_1) is a weak solution of (1.1). Moreover, the fact $(u_n, v_n) \in \mathcal{N}_{\lambda,\mu}$ implies that

$$Q_{\lambda,\mu}(u_n, v_n) = \frac{q(2_s^*(\beta) - 2)}{2(2_s^*(\beta) - q)} \|(u_n, v_n)\|_W^2 - \frac{q 2_s^*(\beta)}{2_s^*(\beta) - q} I_{\lambda,\mu}(u_n, v_n). \tag{5.4}$$

Let $n \to \infty$ in (5.4), by (5.3) and the fact that $c_{\lambda,\mu} < 0$, we obtain

$$Q_{\lambda,\mu}(u_1, v_1) \geq -\frac{q 2_s^*(\beta)}{2_s^*(\beta) - q} c_{\lambda,\mu} > 0.$$

Thus, $(u_1, v_1) \in \mathcal{N}_{\lambda,\mu}$ is a nontrivial weak solution of (1.1).

Next, we prove that $(u_n, v_n) \to (u_1, v_1)$ strongly in W and $I_{\lambda,\mu}(u_1, v_1) = c_{\lambda,\mu}$. From the fact $(u_1, v_1) \in \mathcal{N}_{\lambda,\mu}$ and the Fatou's lemma, it follows that

$$\begin{aligned} c_{\lambda,\mu} \leq I_{\lambda,\mu}(u_1, v_1) &= \frac{2_s^*(\beta) - 2}{2\, 2_s^*(\beta)} \|(u_1, v_1)\|_W^2 - \frac{2_s^*(\beta) - q}{q 2_s^*(\beta)} Q_{\lambda,\mu}(u_1, v_1) \\ &\leq \lim_{n\to\infty} \left[\frac{2_s^*(\beta) - 2}{2\, 2_s^*(\beta)} \|(u_n, v_n)\|_W^2 - \frac{2_s^*(\beta) - q}{q 2_s^*(\beta)} Q_{\lambda,\mu}(u_n, v_n) \right] \\ &= \lim_{n\to\infty} I_{\lambda,\mu}(u_n, v_n) = c_{\lambda,\mu}, \end{aligned}$$

which implies that $c_{\lambda,\mu} = I_{\lambda,\mu}(u_1, v_1)$ and $\lim_{n\to\infty} \|(u_n, v_n)\|_W^2 = \|(u_1, v_1)\|_W^2$. Standard argument shows that $(u_n, v_n) \to (u_1, v_1)$ strongly in W.

Finally, we claim that $(u_1, v_1) \in \mathcal{N}_{\lambda,\mu}^+$. Otherwise, if $(u_1, v_1) \in \mathcal{N}_{\lambda,\mu}^-$, then by Lemma 8, there exist unique t_1^+ and $t_1^- > 0$ such that $(t_1^+ u_1, t_1^+ v_1) \in \mathcal{N}_{\lambda,\mu}^+$ and $(t_1^- u_1, t_1^- v_1) \in \mathcal{N}_{\lambda,\mu}^-$. In particular, we have $t_1^+ < t_1^- = 1$. Since

$$\frac{d}{dt} I_{\lambda,\mu}(t_1^+ u_1, t_1^+ v_1) = 0, \qquad \frac{d^2}{dt^2} I_{\lambda,\mu}(t_1^+ u_1, t_1^+ v_1) > 0,$$

there exists $t_1^* \in (t_1^+, t_1^-)$ such that $I_{\lambda,\mu}(t_1^+ u_1, t_1^+ v_1) < I_{\lambda,\mu}(t_1^* u_1, t_1^* v_1)$. By Lemma 8, we have

$$I_{\lambda,\mu}(t_1^+ u_1, t_1^+ v_1) < I_{\lambda,\mu}(t_1^* u_1, t_1^* v_1) \leq I_{\lambda,\mu}(t_1^- u_1, t_1^- v_1) = I_{\lambda,\mu}(u_1, v_1),$$

which contradicts $I_{\lambda,\mu}(u_1, v_1) = c_{\lambda,\mu}$. Moreover, from

$$I_{\lambda,\mu}(u_1, v_1) = I_{\lambda,\mu}(|u_1|, |v_1|), \quad (|u_1|, |v_1|) \in \mathcal{N}_{\lambda,\mu}^+$$

and the strong maximum principle [2], we conclude that $u_1,\ v_1 > 0$. Hence, (u_1, v_1) is a positive solution for (1.1). □

Proof of Theorem 2 (i). By Theorem 3, we obtain that for all $\lambda,\ \mu > 0$ with $0 < (\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$, problem (1.1) has a positive solution $(u_1, v_1) \in \mathcal{N}_{\lambda,\mu}^+$. □

Remark 2. From Lemma 7 (i) and (4.2), for this positive solution (u_1, v_1), we have

$$
\begin{aligned}
0 > c_{\lambda,\mu} = I_{\lambda,\mu}(u_1, v_1) &= \Big(\frac{1}{2} - \frac{1}{2_s^*(\beta)}\Big)\|(u_1, v_1)\|_W^2 \\
&- \Big(\frac{1}{q} - \frac{1}{2_s^*(\beta)}\Big) Q_{\lambda,\mu}(u_1, v_1) \geq -\frac{2_s^*(\beta) - q}{q\, 2_s^*(\beta)} Q_{\lambda,\mu}(u_1, v_1) \\
&\geq -\frac{2_s^*(\beta) - q}{q\, 2_s^*(\beta)} \Lambda(s, \alpha)^{-\frac{q}{2}} \|(u_1, v_1)\|_W^q \\
&\times \Big((\lambda \|f\|_{L^p(\Omega, |x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega, |x|^{-\alpha})})^{\frac{2}{2-q}} \Big)^{\frac{2-q}{2}}.
\end{aligned}
$$

This implies that $I_{\lambda,\mu}(u_1, v_1) \to 0$ as $\lambda \to 0^+$ and $\mu \to 0^+$.

Next, we establish the existence of a local minimum for $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}^-$.

Lemma 10. *Under the assumptions of Theorem 2, there exist a nonnegative function* $(u_0, v_0) \in W \backslash \{(0, 0)\}$ *and* $\widehat{\Lambda} > 0$ *such that*

$$
\sup_{t \geq 0} I_{\lambda,\mu}(tu_0, tv_0) < c_\infty \tag{5.5}
$$

for all λ, $\mu > 0$ *with* $0 < (\lambda \|f\|_{L^p(\Omega, |x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega, |x|^{-\alpha})})^{\frac{2}{2-q}} < \widehat{\Lambda}$, *where* c_∞ *is the constant given in* (3.3). *In particular,* $c_{\lambda,\mu}^- < c_\infty$.

Proof. Now, we first consider the functional $J : W \to \mathbb{R}$ defined by

$$
J(u, v) = \frac{1}{2}\|(u, v)\|_W^2 - \frac{1}{2_s^*(\beta)} \int_\Omega \frac{F(u, v)}{|x|^\beta} dx.
$$

From Lemma 1, there exists $(e_1, e_2) \in \mathbb{R}^2$ such that $e_1^2 + e_2^2 = 1$ and $M_F = F(e_1, e_2)^{\frac{2}{2_s^*(\beta)}}$. Set $u_0 = e_1 u_\varepsilon$, $v_0 = e_2 u_\varepsilon$, where $u_\varepsilon(x) = \varphi(x) U_\varepsilon(x)$, $\varepsilon > 0$, given by (2.10). Then, by $S_F(s, \beta) = M_F^{-1} \Lambda(s, \beta)$, (2.11), (2.12) and the fact

$$
\max_{t \geq 0} \left(\frac{t^2}{2} B_1 - \frac{t^{2_s^*(\beta)}}{2_s^*(\beta)} B_2 \right) = \frac{2_s^*(\beta) - 2}{2 2_s^*(\beta)} \left(B_1\, B_2^{-\frac{2}{2_s^*(\beta)}} \right)^{\frac{2_s^*(\beta)}{2_s^*(\beta) - 2}}, \quad B_1\ B_2 > 0,
$$

we conclude that

$$
\begin{aligned}
\sup_{t \geq 0} J(tu_0, tv_0) &\leq \frac{2_s^*(\beta) - 2}{2 2_s^*(\beta)} \left(\frac{(e_1^2 + e_2^2)\|u_\varepsilon\|_\gamma^2}{\left(\int_\Omega \frac{F(e_2 u_\varepsilon, e_2 u_\varepsilon)}{|x|^\beta} dx \right)^{\frac{2}{2_s^*(\beta)}}} \right)^{\frac{2_s^*(\beta)}{2_s^*(\beta) - 2}} \\
&= \frac{2_s^*(\beta) - 2}{2 2_s^*(\beta)} \left(\frac{(e_1^2 + e_2^2)\|u_\varepsilon\|_\gamma^2}{\left(M_F^{\frac{2_s^*(\beta)}{2}} \int_\Omega \frac{|u_\varepsilon|^{2_s^*(\beta)}}{|x|^\beta} dx \right)^{\frac{2}{2_s^*(\beta)}}} \right)^{\frac{2_s^*(\beta)}{2_s^*(\beta) - 2}} \leq \frac{2_s^*(\beta) - 2}{2 2_s^*(\beta)} \left(\frac{1}{M_F} \right)^{\frac{2_s^*(\beta)}{2_s^*(\beta) - 2}}
\end{aligned}
$$

$$
\times\left[\frac{\Lambda(s,\beta)^{\frac{2_s^*(\beta)}{2_s^*(\beta)-2}}+O(\varepsilon^{2b(\gamma)+2s-N})}{\left(\Lambda(s,\beta)^{\frac{2_s^*(\beta)}{2_s^*(\beta)-2}}+O(\varepsilon^{2_s^*(\beta)b(\gamma)+\beta-N})\right)^{\frac{2}{2_s^*(\beta)}}}\right]^{\frac{2_s^*(\beta)}{2_s^*(\beta)-2}}
\leq \frac{2s-\beta}{2(N-\beta)}S_F(s,\beta)^{\frac{2_s^*(\beta)}{2_s^*(\beta)-2}}+O(\varepsilon^{2b(\gamma)+2s-N}). \tag{5.6}
$$

Let C_0 be a positive constant given in Lemma 2, we can choose $\Lambda_2>0$ small enough such that $c_\infty>0$ for all λ, $\mu>0$ with $0<(\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}+(\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}<\Lambda_2$. Using the definition of $I_{\lambda,\mu}(u,v)$, we have

$$
I_{\lambda,\mu}(tu_0,tv_0)\leq\frac{t^2}{2}\|(u_0,v_0)\|_W^2\leq Ct^2,\quad\forall t\geq 0,\ \lambda,\ \mu>0,
$$

which implies that there exists $t_0\in(0,1)$ such that

$$
\sup_{t\in[0,t_0]} I_{\lambda,\mu}(tu_0,tv_0)<c_\infty \tag{5.7}
$$

for all λ, $\mu>0$ with $0<(\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}+(\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}<\Lambda_2$.

Next, we prove that $\sup\limits_{t\in[t_0,\infty)} I_{\lambda,\mu}(tu_0,tv_0)<c_\infty$. Since $f(x)$, $g(x)\geq a_0$ for all $x\in B_{r_0}(0)\subset\Omega$, we have

$$
\begin{aligned}
Q_{\lambda,\mu}(u_0,v_0)&=\lambda e_1^q\int_\Omega\frac{f(x)|u_\varepsilon|^q}{|x|^\alpha}dx+\mu e_2^q\int_\Omega\frac{g(x)|u_\varepsilon|^q}{|x|^\alpha}dx\\
&\geq a_0M(\lambda+\mu)\int_{B_\rho(0)}\frac{|u_\varepsilon|^q}{|x|^\alpha}dx,
\end{aligned}\tag{5.8}
$$

where $M=\min\{e_1^q,e_2^q\}$. Combining (5.6), (5.8) and (2.13), for all $t\geq t_0$, we get

$$
\begin{aligned}
&\sup_{t\geq t_0} I_{\lambda,\mu}(tu_0,tv_0)=\sup_{t\geq t_0}\Big(J(tu_0,tv_0)-\frac{t^q}{q}Q_{\lambda,\mu}(u_0,v_0)\Big)\\
&\leq\frac{2s-\beta}{2(N-\beta)}S_F(s,\beta)^{\frac{2_s^*(\beta)}{2_s^*(\beta)-2}}+O(\varepsilon^{2b(\gamma)+2s-N})-\frac{t_0^q}{q}a_0M(\lambda+\mu)\int_{B_\rho(0)}\frac{|u_\varepsilon|^q}{|x|^\alpha}dx\\
&\leq\frac{2s-\beta}{2(N-\beta)}S_F(s,\beta)^{\frac{2_s^*(\beta)}{2_s^*(\beta)-2}}+O(\varepsilon^{2b(\gamma)+2s-N})\\
&\qquad-(\lambda+\mu)\begin{cases}C\varepsilon^{N-\alpha-\frac{q(N-2s)}{2}}, & \text{if}\quad q>(N-\alpha)/b(\gamma),\\ C\varepsilon^{N-\alpha-\frac{q(N-2s)}{2})}|\ln\varepsilon|, & \text{if}\quad q=(N-\alpha)/b(\gamma),\\ C\varepsilon^{q(b(\gamma)-\frac{N-2s}{2})}, & \text{if}\quad q<(N-\alpha)/b(\gamma).\end{cases}
\end{aligned}\tag{5.9}
$$

(i) If $1\leq q<\frac{N-\alpha}{b(\gamma)}$. Since $b(\gamma)>\frac{N-2s}{2}$, we get $2b(\gamma)+2s-N>q(b(\gamma)-\frac{N-2s}{2})$. Then, for ε small enough, we can choose $\Lambda_3>0$ such that

$$
\begin{aligned}
&O(\varepsilon^{2b(\gamma)+2s-N})-C(\lambda+\mu)\varepsilon^{q(b(\gamma)-\frac{N-2s}{2})}\\
&\qquad<-C_0\left((\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}+(\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}\right)
\end{aligned}\tag{5.10}
$$

for all $\lambda,\ \mu > 0$ with $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_3$. Set $\Lambda_4 = \min\{\Lambda_2, \Lambda_3\}$, then for all $\lambda,\ \mu > 0$ with $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_4$, (5.7), (5.9) and (5.10) show that

$$\sup_{t\geq 0} I_{\lambda,\mu}(tu_0, tv_0) < c_\infty.$$

(ii) If $\frac{N-\alpha}{b(\gamma)} \leq q < 2$. From $b(\gamma) > \frac{N-2s}{2}$ and $\frac{N-\alpha}{b(\gamma)} \leq q$, we can obtain $2b(\gamma) + 2s - N > N - \alpha - \frac{q(N-2s)}{2}$. Then, for ε small enough, there exists a $\Lambda_5 > 0$ such that

$$\begin{aligned} O(\varepsilon^{2b(\gamma)+2s-N}) &- C(\lambda+\mu)\varepsilon^{N-\alpha-\frac{q(N-2s)}{2}} \\ &< -C_0\Big((\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}\Big) \end{aligned} \tag{5.11}$$

for all $\lambda,\ \mu > 0$ with $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_5$. Similarly, let $\Lambda_6 = \min\{\Lambda_2, \Lambda_5\}$, by (5.7), (5.9) and (5.11), one can get

$$\sup_{t\geq 0} I_{\lambda,\mu}(tu_0, tv_0) < c_\infty$$

for all $\lambda,\ \mu > 0$ with $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_6$.

Set $\widehat{\Lambda} = \min\{\Lambda_4, \Lambda_6\}$, from cases (i) and (ii), (5.5) holds by taking $(u_0, v_0) = (e_1\, u_\varepsilon, e_2\, u_\varepsilon)$ and for all $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \widehat{\Lambda}$.

Recalling that $(u_0, v_0) = (e_1\, u_\varepsilon, e_2\, u_\varepsilon)$, it is easy to see that

$$\int_\Omega \frac{F(u_0, v_0)}{|x|^\beta}dx > 0 \quad \text{and} \quad Q_{\lambda,\mu}(u_0, v_0) > 0. \tag{5.12}$$

Then, from (5.12) and Lemma 8, we get that there exists $t^- > 0$ such that $(t^-u_0, t^-v_0) \in \mathcal{N}^-_{\lambda,\mu}$. Thus, it follows from the definition of $c^-_{\lambda,\mu}$ and (5.5) that $c^-_{\lambda,\mu} \leq I_{\lambda,\mu}(t^-u_0, t^-v_0) \leq \sup_{t\geq 0} I_{\lambda,\mu}(tu_0, tv_0)) < c_\infty$ for all $\lambda,\ \mu > 0$ with $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \widehat{\Lambda}$. The proof is thus complete. □

Now, we establish the existence of a local minimum of $I_{\lambda,\mu}$ on $\mathcal{N}^-_{\lambda,\mu}$.

Theorem 4. *Set* $\Lambda^* = \min\{\Lambda_1, \widehat{\Lambda}\}$, *under the assumptions of Theorem 2, the problem* (1.1) *has a positive solution* $(u_2, v_2) \in \mathcal{N}^-_{\lambda,\mu}$ *and* $I_{\lambda,\mu}(u_2, v_2) = c^-_{\lambda,\mu}$ *for all* $\lambda,\ \mu > 0$ *with* $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda^*$.

Proof. Set $\Lambda^* = \min\{\Lambda_1, \widehat{\Lambda}\}$. By Proposition 9 (ii), Lemmas 3 and 10, for all $\lambda,\ \mu > 0$ with $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda^*$, $I_{\lambda,\mu}$ satisfies the $(PS)_{c^-_{\lambda,\mu}}$ condition for all $c^-_{\lambda,\mu} \in (0, c_\infty)$. Since $I_{\lambda,\mu}$ is coercive on $\mathcal{N}^-_{\lambda,\mu}$, we get that the $(PS)_{c^-_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\}$ is bounded. Therefore, there exist a subsequence still denoted by $\{(u_n, v_n)\}$ and $(u_2, v_2) \in W\backslash\{(0,0)\}$

such that $(u_n, v_n) \rightharpoonup (u_2, v_2)$ weakly in W. Arguing as in the proof of Theorem 3, we obtain $(u_n, v_n) \to (u_2, v_2)$ strongly in W and (u_2, v_2) is a positive solution of (1.1) for all λ, $\mu > 0$ with $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda^*$.

Finally, we prove that $(u_2, v_2) \in \mathcal{N}^-_{\lambda,\mu}$. Arguing by contradiction, we assume $(u_2, v_2) \in \mathcal{N}^+_{\lambda,\mu}$. Since $\mathcal{N}^-_{\lambda,\mu}$ is closed in W, we have $\|(u_2, v_2)\|_W < \liminf_{n\to\infty} \|(u_n, v_n)\|_W$. Moreover, by Lemma 8, there exists a unique t_2^- such that $(t_2^- u_2, t_2^- v_2) \in \mathcal{N}^-_{\lambda,\mu}$. This and $(u_n, v_n) \in \mathcal{N}^-_{\lambda,\mu}$ deduce that

$$c^-_{\lambda,\mu} \le I_{\lambda,\mu}(t_2^- u_2, t_2^- v_2) < \lim_{n\to\infty} I_{\lambda,\mu}(t_2^- u_n, t_2^- v_n) \le \lim_{n\to\infty} I_{\lambda,\mu}(u_n, v_n) = c^-_{\lambda,\mu}.$$

So, $(u_2, v_2) \in \mathcal{N}^-_{\lambda,\mu}$. This completes the proof of Theorem 4. □

Proof of Theorem 2 (ii). By Theorem 3, the system (1.1) has a positive solution $(u_1, v_1) \in \mathcal{N}^+_{\lambda,\mu}$ for all λ, $\mu > 0$ with $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$. On the other hand, from Theorem 4, we can get the second positive solution $(u_2, v_2) \in \mathcal{N}^-_{\lambda,\mu}$ for all λ, $\mu > 0$ with $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda^*$. Since $\mathcal{N}^+_{\lambda,\mu} \cap \mathcal{N}^-_{\lambda,\mu} = \emptyset$ and $\Lambda^* < \Lambda_0$, we get (u_1, v_1), (u_2, v_2) are distinct positive solutions of (1.1) for all λ, $\mu > 0$ with $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda^*$. This completes the proof of Theorem 2 (ii). □

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