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QUASI-INTERPOLATION BY SPLINES ON THE UNIFORM KNOT SETS

E. LEETMA and G. VAINIKKO

University of Tartu

Liivi 2, 50409 Tartu, Estonia

E-mail: `evely.leetma@ut.ee; gennadi.vainikko@ut.ee`

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Abstract. In the case of uniform grids, the error of the spline interpolant of a function defined on $\mathbb{R}$ has been well estimated. On the basis of the spline interpolation formula for functions defined on $\mathbb{R}$ we derive quasi-interpolation formulae for functions defined on $\mathbb{R}$ or in a vicinity of a bounded interval, say $[0, 1]$, and we estimate the difference between the interpolant and the quasi-interpolants.

Key words: splines, interpolation, quasi-interpolation, weakly singular integral equations

1. Introduction

Our problem setting origins from the collocation type methods to solve weakly singular integral equations

$$v(t) = \int_0^1 K(t,s)v(s)ds + f(t), \qquad 0 \le t \le 1,$$

where $f \in C^m[0,1]$ and K is C^m-smooth on $[0,1] \times [0,1] \setminus \text{diag}$ having an integrable singularity on the diagonal. Then a solution $v \in C[0,1]$ is C^m-smooth in the interval $(0,1)$ but its derivatives typically have boundary singularities. With a suitable change of variables $t = \varphi(x)$, $s = \varphi(y)$, the integral equation can be transformed so that the singularities of the solution $u(x) := v(\varphi(x))$ will be suppressed (see [3] for details) and, moreover, $u^{(j)}(0) = u^{(j)}(1) = 0,\ j = 1, \dots, m$. There is a simple extension $\overline{u}$ onto $\mathbb{R}$ of such u defined by $\overline{u}(x) = u(0)$ for $x < 0$ and $\overline{u}(x) = u(1)$ for $x > 1$. This extension preserves the smoothness and bounds for the derivatives. Alternatively, we can decompose $u(x) = \widetilde{u}(x) + [u(1) - u(0)]x$ for $0 \le x \le 1$, then $\widetilde{u}(x) = u(x) - [u(1) - u(0)]x$ has a C^m-smooth 1-periodic extension from $[0,1]$ onto $\mathbb{R}$ and $\widetilde{u}^{(m)}(x) = u^{(m)}(x)$ for $0 \le x \le 1$. This enables to replace the interpolation/quasi-interpolation of $u \in C[0,1]$ by the interpolation/quasi-interpolation of $\overline{u} \in BC(\mathbb{R})$ or $\widetilde{u} \in C_{\text{per}}(\mathbb{R})$ on $\mathbb{R}$ and build corresponding collocation/quasi-collocation methods for integral equations. Designing fast solvers, we are strongly interested in smallest constants in error estimates for interpolation/quasi-interpolation.

In the present paper we construct $2p-1$ point spline quasi-interpolants $Q_{h,m}^{(p)} f$, $p \in \mathbb{N}$, for functions f given in a vicinity of the standard interval $[0,1]$, starting from the formula for the spline interpolant $Q_{h,m}\bar{f}$ of order m (or, of degree $m-1$) with the uniform knot set of a step size $h > 0$. Here $\bar{f}$ is a special extension of f onto $\mathbb{R}$. An optimal error estimate of $\bar{f} - Q_{h,m}\bar{f}$ (with smallest possible constant) on the Sobolev classes $W^{m,\infty}(\mathbb{R})$ and $V^{m,\infty}(\mathbb{R})$ is known, so it remains to estimate $Q_{h,m}\bar{f} - Q_{h,m}^{(p)} f$ on $[0,1]$. For $p \geq p' = \text{int}((m+2)/2)$, the error $Q_{h,m}\bar{f} - Q_{h,m}^{(p)} f$ occurs to be asymptotically smaller than the error of $\bar{f} - Q_{h,m}\bar{f}$ (provided that f is not a polynomial of degree $m-1$), and then the main part of the error $f - Q_{h,m}^{(p)} f$ is generated by $\bar{f} - Q_{h,m}\bar{f}$.

So we proceed in the inverse direction compared to [1, 4] where first an error estimate of the quasi-interpolant was derived directly and then used to estimate the error of the interpolant; this latter way enables results of optimal accuracy order but with strongly overestimated (or undetermined) constants in the error estimates.

We also discuss the operator norms of $Q_{h,m}$ and $Q_{h,m}^{(p)}$ in the space $BC(\mathbb{R})$ of bounded continuous functions u on $\mathbb{R}$ equipped with the norm

$$\|u\|_\infty = \sup_{x \in \mathbb{R}} |u(x)|.$$

The numerical values of norms $\|Q_{h,m}\|$ and $\|Q_{h,m}^{(p)}\|$ for moderate m occur to be surprisingly small, e.g., $\|Q_{h,m}\| = 2.142$, $\|Q_{h,m}^{(p')}\| = 1.419$ for $m = 10$, in contrast to extremely pessimistic estimate $\|Q_{h,m}^{(p')}\| \leq (2m)^m$ in [4]; it must be said that the spline grids may be non-uniform in [1, 4], and in the case of uniform grids, the quasi-interpolants of [1, 4] are different from our ones. It seems that $\|Q_{h,m}\|$ grows logarithmically as $m \to \infty$ but we have no analytic proof of this empiric guess.

2. Cardinal B-Splines

We present two equivalent definitions of the *cardinal* B-*spline* B_m of *order* m in terminology of [1, 4], or of degree $m-1$ in terminology of [2, 5, 7].

DEFINITION 1 (explicit formula):

$$B_m(x) = \frac{1}{(m-1)!} \sum_{i=0}^{m} (-1)^i \binom{m}{i} (x-i)_+^{m-1}, \quad x \in \mathbb{R}, \quad m \in \mathbb{N},$$

where, as usual, $0! = 1$, $0^0 := \lim_{x \downarrow 0} x^x = 1$,

$$\binom{m}{i} = \frac{m!}{i!\,(m-i)!}, \quad (x-i)_+^{m-1} := \left\{ \begin{matrix} (x-i)^{m-1}, & x-i \geq 0 \\ 0, & x-i < 0 \end{matrix} \right\}.$$

DEFINITION 2 (recursive formula):

$$B_1(x) = \left\{ \begin{matrix} 1, & 0 \leq x < 1 \\ 0, & x \in \mathbb{R} \setminus [0,1) \end{matrix} \right\}, \quad B_m(x) = \int_{x-1}^{x} B_{m-1}(y)\,dy, \ \ m = 2, 3, \ldots .$$

Let us list some properties of the cardinal B-splines:

$$B_m\,|_{[i,i+1]} \in \mathcal{P}_{m-1} \text{ (polynomials of degree } m-1), \ \ i \in \mathbb{Z}, \quad B_m \in C^{(m-2)}(\mathbb{R}),$$

i.e., B_m is a spline of defect 1, degree $m-1$ on the "cardinal" knot set $\mathbb{Z}$,

$$B_m^{(m-1)}(x) = (-1)^i \binom{m-1}{i} \text{ for } i < x < i+1, \ \ i = 0, \ldots, m-1,$$

$$\text{supp}B_m = [0,m], \quad B_m(x) > 0 \text{ for } 0 < x < m,$$

$$B_m\left(\frac{m}{2} - x\right) = B_m\left(\frac{m}{2} + x\right), \ \ x \in \mathbb{R}, \quad B_m\left(\frac{m}{2}\right) = \max_{x \in \mathbb{R}} B_m(x),$$

$$\int_{\mathbb{R}} B_m(x)dx = 1, \quad \sum_{j \in \mathbb{Z}} B_m(x-j) = 1, \ \ x \in \mathbb{R}.$$

3. The Wiener Interpolant

Assume $m \geq 3$ to be fixed. Introduce the knot set $\{jh : j \in \mathbb{Z}\}$, $h > 0$, of splines and the set of interpolation knots $\{(k + \frac{m}{2})h, \, k \in \mathbb{Z}\}$. Given a bounded or polynomially growing function $f \in C(\mathbb{R})$, we look for its interpolant $Q_{h,m}f$ in the form

$$(Q_{h,m}f)(x) = \sum_{j \in \mathbb{Z}} d_j B_m\left(\frac{x}{h} - j\right), \ \ x \in \mathbb{R}, \tag{3.1}$$

and determine its coefficients d_j by the interpolation conditions

$$(Q_{h,m}f)\left(\left(k + \frac{m}{2}\right)h\right) = f\left(\left(k + \frac{m}{2}\right)h\right), \ \ k \in \mathbb{Z}. \tag{3.2}$$

Conditions (3.1) and (3.2) lead to the bi-infinite system of linear equations

$$\sum_{j \in \mathbb{Z}} b_{k-j} d_j = f_k, \quad k \in \mathbb{Z}, \tag{3.3}$$

where

$$b_k = b_{k,m} = B_m\left(k + \frac{m}{2}\right), \ \ f_k = f_{k,h,m} = f\left(\left(k + \frac{m}{2}\right)h\right), \ \ k \in \mathbb{Z}, \tag{3.4}$$

$$b_k = b_{-k} > 0 \ \text{for} \ |k| \le \mu, \ \ b_k = 0 \ \text{for} \ |k| > \mu, \ \ \sum_{|k|\le\mu} b_k = 1,$$

$$\mu := \text{int}((m-1)/2) = \left\{ \begin{array}{l} (m-2)/2, \ m \text{ even} \\ (m-1)/2, \ m \text{ odd} \end{array} \right\} \quad (\text{int} = \text{integer part}).$$

Thus (3.3) is a symmetric bi-infinite band system with the band width $2\mu+1$. The solution of system (3.3) exists but is nonunique if we do not set the restriction that $|d_j|$ are bounded or of a polynomial growth as $|j| \to \infty$. The only reasonable solution of (3.3) is based on the Wiener theorem about the trigonometric series which we reformulate for the Laurent series as follows:

Given (possibly complex) numbers b_k, $k \in \mathbb{Z}$, such that

$$\sum_{k\in\mathbb{Z}} |b_k| < \infty, \ \ b(z) := \sum_{k\in\mathbb{Z}} b_k z^k \neq 0 \ \text{ for all } \ z \in \mathbb{C} \text{ with } |z| = 1, \tag{3.5}$$

the expansion $a(z) := 1/b(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ satisfies $\sum_{k\in\mathbb{Z}} |a_k| < \infty$.

By the Wiener solution of the system $\sum_{j\in\mathbb{Z}} b_{k-j} d_j = f_k, \ \ k \in \mathbb{Z}$, we mean $d_k = \sum_{j\in\mathbb{Z}} a_{k-j} f_j, \ \ k \in \mathbb{Z}$.

With $b_k = b_{k,m}$ defined in (3.4), introduce the functions

$$b(z) = b^m(z) := \sum_{|k|\le\mu} b_k z^k = b_0 + \sum_{k=1}^{\mu} b_k (z^k + z^{-k}), \ \ 0 \neq z \in \mathbb{C}, \tag{3.6}$$

$$P_{2\mu}(z) = P_{2\mu}^m(z) = z^\mu b^m(z) \ \ (\text{the characteristic polynomial of } B_m), \tag{3.7}$$

$$a(z) = a^m(z) := 1/b^m(z) = z^\mu / P_{2\mu}^m(z), \ \ z \in \mathbb{C}, \ \ z \neq z_\nu, \ \nu = 1, \ldots, 2\mu, \tag{3.8}$$

where z_ν, $\nu = 1, \ldots, 2\mu$, are the roots of $P_{2\mu}^m \in \mathcal{P}_{2\mu}$, called the *characteristic roots*. From (3.6)–(3.7) we observe that together with z_ν also $1/z_\nu$ is a characteristic root. The polynomials $P_{2\mu}^m(z)$ were introduced in [5] starting from other considerations, and it was stated in [5] that *the characteristic roots are real and simple*; then clearly $z_\nu < 0$, $\nu = 1, \ldots, 2\mu$ and $z_\nu \neq -1$, $\nu = 1, \ldots, 2\mu$, thus there is exactly μ characteristic roots in the interval $(-1, 0)$ and the remaining μ ones are in $(-\infty, -1)$; in particular, conditions (3.5) are fulfilled. For instance, for $m = 10$ (then $\mu = 4$) we have

$$P_8^{10}(z) = \frac{1}{9!}\left[(z^8+1) + 502(z^7+z) + 14608(z^6+z^2) + 88234(z^5+z^3) + 156190z^4\right],$$

$$z_1 = -0.002121, \ \ z_2 = -0.043223, \ \ z_3 = -0.201751, \ \ z_4 = -0.607997,$$

$$z_5 = 1/z_1, \quad z_6 = 1/z_2, \quad z_7 = 1/z_3, \quad z_8 = 1/z_4.$$

The coefficients $a_k = a_{k,m}$ of the expansion $a(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ can be expressed through characteristic roots $z_\nu \in (-1, 0)$, $\nu = 1, \ldots \mu$, by the formula (cf. [5])

$$a_k = \sum_{\nu=1}^{\mu} \frac{z_\nu^{\mu-1}}{P_{2\mu}'(z_\nu)} z_\nu^{|k|}, \ \ k \in \mathbb{Z}; \ \ \sum_{k\in\mathbb{Z}} a_k = 1, \ \ \sum_{k\in\mathbb{Z}} |a_k| = \frac{(-1)^\mu}{P_{2\mu}(-1)}. \tag{3.9}$$

Thus we have in hand the *Wiener interpolant* $Q_{h,m}f$ defined by

$$(Q_{h,m}f)(x) = \sum_{k\in\mathbb{Z}} d_k B_m(h^{-1}x - k), \quad x \in \mathbb{R}, \quad d_k = \sum_{j\in\mathbb{Z}} a_{k-j} f_j, \quad k \in \mathbb{Z}.$$

Clearly a_k are real, $a_k = a_{-k}$, $k \in \mathbb{Z}$, and a_k decays exponentially as $|k| \to \infty$.

Introducing the *fundamental spline* $F_m(x) := \sum_{j\in\mathbb{Z}} a_j B_m(x - j)$ (it satisfies $F_m(i + \frac{m}{2}) = \delta_{i,0}$, $i \in \mathbb{Z}$) and denoting $\varphi_m(x) = \sum_{k\in\mathbb{Z}} |F_m(x - k)|$ (it is an 1-periodic function with the property $\varphi_m(\frac{m}{2} - x) = \varphi_m(\frac{m}{2} + x)$), it is easily seen that

$$q_m := \|Q_{n,m}\|_{BC(\mathbb{R})\to BC(\mathbb{R})} = \max_{x\in\mathbb{R}} \varphi_m(x) = \max_{\frac{m}{2}\le x\le\frac{m+1}{2}} \varphi_m(x) \le \sum_{k\in\mathbb{Z}} |a_k|.$$

For $m \le 20$, the interpolation process has good stability properties:

Table 1. Numerical values of q_m and $\sum_k |a_{k,m}|$.

m	3	4	5	6	7	8	9	10	20
q_m	1.414	1.549	1.706	1.816	1.916	2.000	2.075	2.142	2.583
$\sum_k \lvert a_{k,m}\rvert$	2.000	3.000	4.800	7.500	11.80	18.53	29.11	45.73	4182

For $4 \le m \le 20$, q_m fits into the model $q_m \le \frac{e}{4} + \frac{2}{\pi}\log m$, and possibly $q_m - (\frac{e}{4} + \frac{2}{\pi}\log m) \to 0$ as $m \to \infty$; for $m = 20$ this difference is 0.0036. We can also observe that $\sum_k |a_{k,m+1}| / \sum_k |a_{k,m}| \to \pi/2 = 1.5707963268\ldots$ as $m \to \infty$; for $m = 20$ this ratio is 1.570796327. It is a challenge to confirm these empiric guesses analytically.

In analogy to the Sobolev space $W^{m,\infty}(\mathbb{R})$, introduce the space $V^{m,\infty}(\mathbb{R})$, $m \in \mathbb{N}$, consisting of functions $f \in C^{m-1}(\mathbb{R})$ such that $f^{(m)} \in L^\infty(\mathbb{R})$ (the derivatives are understood in the sense of distributions). A function $f \in V^{m,\infty}(\mathbb{R})$ may grow as $|x| \to \infty$. With the help of the Taylor formula

$$f(x) = \sum_{l=0}^{m-1} \frac{f^{(l)}(0)}{l!} x^l + \frac{1}{(m-1)!} \int_0^x (x-t)^{m-1} f^{(m)}(t)\, dt, \quad x \in \mathbb{R},$$

we observe that

$$|f(x)| \le \|f^{(m)}\|_\infty \frac{1}{m!} |x|^m + O(x^{m-1}) \quad \text{as } |x| \to \infty.$$

Hence, $Q_{h,m}f$ is well defined for $f \in V^{m,\infty}(\mathbb{R})$. Clearly, $W^{m,\infty}(\mathbb{R}) + \mathcal{P}_m \subset V^{m,\infty}(\mathbb{R})$; this inclusion is strict.

Theorem 1. *For $f \in V^{m,\infty}(\mathbb{R})$, $m \in \mathbb{N}$, it holds*

$$\|f - Q_{h,m}f\|_\infty \le \Phi_{m+1} \pi^{-m} h^m \|f^{(m)}\|_\infty, \tag{3.10}$$

where Φ_{m+1} is the Favard constant defined by

$$\Phi_m = \frac{4}{\pi} \begin{cases} \sum\limits_{k=0}^{\infty} \dfrac{1}{(2k+1)^m}, & m = 2l \\ \sum\limits_{k=0}^{\infty} \dfrac{(-1)^k}{(2k+1)^m}, & m = 2l+1 \end{cases}, \quad m \in \mathbb{N};$$

$$\Phi_1 = 1, \ \ \Phi_2 = \pi/2, \ \ \Phi_3 = \pi^2/8, \ \ \Phi_4 = \pi^3/24,$$

$$\Phi_1 < \Phi_3 < \Phi_5 < \ldots < \frac{4}{\pi} < \ldots < \Phi_6 < \Phi_4 < \Phi_2, \quad \lim_{m\to\infty} \Phi_m = \frac{4}{\pi}.$$

For 1-periodic functions f and $h = 1/n$ with even $n \in \mathbb{N}$, Theorem 1 is well known (see [2]), and moreover, estimate (3.10) is then optimal in the sense of Kolmogorov n-width. Namely, for even n, the Kolmogorov n-width of the set $\{f \in W^m_{\text{per}}(\mathbb{R}) : \|f^{(m)}\|_\infty \le 1\}$ is equal to $\Phi_{m+1}\pi^{-m}n^{-m}$, see [2].

A complete proof of Theorem 1 and some further estimates (case of less smooth f, estimates for derivatives) are presented in [6].

Remark 1. Compared with other possible approximations of functions f from values on the uniform grid $\Delta_h = \{(k + \frac{m}{2})h), \ k \in \mathbb{Z}\}$, $Q_{h,m}f$ yields the best approximation on the classes $V^{m,\infty}(\mathbb{R})$ and $W^{m,\infty}(\mathbb{R})$. Namely, for a given positive γ, there is a special function $g \in W^{m,\infty}(\mathbb{R})$, $\|g^{(m)}\|_\infty = \gamma$, such that, for any mapping $M_h : C(\Delta_h) \to C(\mathbb{R})$ (linear or nonlinear, continuous or discontinuous), it holds (cf. (3.10))

$$\max\{\|g - M_h(g\,|_{\Delta_h})\|_\infty, \ \|(-g) - M_h(-g\,|_{\Delta_h})\|_\infty\} \ge \Phi_{m+1}\pi^{-m}h^m\gamma.$$

4. Quasi-Interpolants

Thus, the Wiener interpolant $Q_{h,m}f$ of $f \in C(\mathbb{R})$ is given by

$$(Q_{h,m}f)(x) = \sum_{k\in\mathbb{Z}} \Big(\sum_{j\in\mathbb{Z}} a_{j,m} f_{k-j} \Big) B_m\big(\frac{x}{h} - k\big), \quad a_{j,m} = \sum_{\nu=1}^{\mu} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} z_\nu^{|j|}, \ j \in \mathbb{Z},$$

where $f_k = f((k + \frac{m}{2})h)$, $k \in \mathbb{Z}$. For $m \ge 3$ we approximate $Q_{h,m}f$ by the $2p-1$ point ***quasi-interpolants*** of the form

$$(Q^{(p)}_{h,m}f)(x) = \sum_{k\in\mathbb{Z}} \Big(\sum_{|j|\le p-1} a^{(p)}_{j,m} f_{k-j} \Big) B_m\big(\frac{x}{h} - k\big), \quad p \in \mathbb{N}, \tag{4.1}$$

determining the coefficients $a^{(p)}_{j,m}$ from $a_{j,m}$ with the help of a special difference calculus.

Introduce the vector space $\mathfrak{s}(\mathbb{Z})$ of bisequences $\underline{a} = (a_j)_{j\in\mathbb{Z}}$ such that

$$\forall r \ge 0 \ \exists c_r < \infty : \quad |a_j| \le c_r |j|^{-r}, \ \ 0 \ne j \in \mathbb{Z},$$

and its subspace

$$\mathfrak{s}_{\text{sym}}(\mathbb{Z}) = \{\underline{a} \in \mathfrak{s}(\mathbb{Z}) : \ a_{-j} = a_j, \ j \in \mathbb{Z}\} \subset \mathfrak{s}(\mathbb{Z}).$$

Consider the difference operators

$$D^+ : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z}), \quad (D^+\underline{a})_j = a_{j+1} - a_j, \ j \in \mathbb{Z} \ \text{(forward difference)},$$

$$D^- : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z}), \quad (D^-\underline{a})_j = a_j - a_{j-1}, \ j \in \mathbb{Z} \ \text{(backward difference)}$$

and their one side inverses $J^\pm : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z})$, defined for $k \in \mathbb{Z}$ by

$$(J^+\underline{a})_k = \left\{ \begin{array}{ll} \sum\limits_{j=-\infty}^{k-1} a_j, & k \le 0 \\ -\sum\limits_{j=k}^{\infty} a_j, & k > 0 \end{array} \right\}, \quad (J^-\underline{a})_k = \left\{ \begin{array}{ll} \sum\limits_{j=-\infty}^{k} a_j, & k < 0 \\ -\sum\limits_{j=k+1}^{\infty} a_j, & k \ge 0 \end{array} \right\}.$$

Namely, denoting $\underline{e} = (\delta_{j,0})_{j\in\mathbb{Z}} = (\ldots, 0, 0, 1, 0, 0 \ldots)$, it is easy to check that

$$J^\pm D^\pm \underline{a} = \underline{a}, \quad D^\pm J^\pm \underline{a} = \underline{a} - \Big(\sum_{j\in\mathbb{Z}} a_j\Big)\underline{e}, \quad \text{for } \underline{a} \in \mathfrak{s}(\mathbb{Z}). \tag{4.2}$$

Our main tool is the second order central difference operator

$$D = D^+D^- = D^-D^+ : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z}), \quad (D\underline{a})_j = a_{j-1} - 2a_j + a_{j+1}, \ j \in \mathbb{Z},$$

with its one side inverse

$$J = J^+J^- : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z}).$$

Formulae (4.2) imply that

$$\underline{a} = \Big(\sum_{j\in\mathbb{Z}} a_j\Big)\underline{e} + DJ\underline{a}, \quad \text{for } \underline{a} \in \mathfrak{s}_{\text{sym}}(\mathbb{Z}),$$

and by induction

$$\underline{a} = \sum_{q=0}^{p-1} \gamma_q D^q \underline{e} + D^p J^p \underline{a}, \ \ \gamma_q = \sum_{j\in\mathbb{Z}} (J^q \underline{a})_j, \ \ \underline{a} \in \mathfrak{s}_{\text{sym}}(\mathbb{Z}), \ \ p \in \mathbb{N}. \tag{4.3}$$

Lemma 1. *Let $\underline{a} = (a_{k,m})_{k\in\mathbb{Z}}$ be defined by* (3.9). *Then* (4.3) *holds with*

$$\gamma_0 = 1, \ \ \gamma_q = \gamma_{q,m} = \sum_{\nu=1}^{\mu} \frac{(1+z_\nu) z_\nu^{\mu+q-1}}{(1-z_\nu)^{2q+1} P'_{2\mu}(z_\nu)}, \ \ q \ge 1. \tag{4.4}$$

Respectively, the coefficients $d_k = \sum_{j\in\mathbb{Z}} a_j f_{k-j}$ of the Wiener interpolant can be represented in the form

$$d_k = f_k + \sum_{q=1}^{p-1} \gamma_q D^q f_k + \delta_k^{(p)} = \sum_{|j|\le p-1} a_j^{(p)} f_{k-j} + \delta_k^{(p)}, \quad k \in \mathbb{Z}, \tag{4.5}$$

where

$$\delta_k^{(p)} = \sum_{j\in\mathbb{Z}} (J^p \underline{a})_{k-j} D^p f_j, \ \ k \in \mathbb{Z}, \tag{4.6}$$

$$a_j^{(p)} = a_{j,m}^{(p)} = \sum_{q=|j|}^{p-1} (-1)^{j+q} \binom{2q}{j+q} \gamma_{q,m}, \ \ |j| \le p-1. \tag{4.7}$$

Proof. According to (3.9) and (4.3), $\gamma_0 = \sum_{j\in\mathbb{Z}} a_j = 1$. Let us establish (4.4) for $q \geq 1$. For the bisequence $\underline{z}^{(\nu)} := (z_\nu^{|j|})_{j\in\mathbb{Z}}$ we have

$$(J^- \underline{z}^{(\nu)})_k = \frac{1}{1-z_\nu}\begin{Bmatrix} z_\nu^{-k}, & k<0 \\ -z_\nu^{k+1}, & k\geq 0 \end{Bmatrix},$$

$$(J\underline{z}^{(\nu)})_k = (J^+J^-\underline{z}^{(\nu)})_k = \frac{1}{(1-z_\nu)^2}\begin{Bmatrix} z_\nu^{-k+1}, & k\leq 0 \\ z_\nu^{k+1}, & k>0 \end{Bmatrix} = \frac{z_\nu}{(1-z_\nu)^2} z_\nu^{|k|}.$$

Thus

$$(J\underline{a})_k = \sum_{\nu=1}^{\mu} \frac{z_\nu}{(1-z_\nu)^2}\,\frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)}\, z_\nu^{|k|}, \quad k\in\mathbb{Z}.$$

By repeated application of this formula we find that

$$(J^q\underline{a})_k = \sum_{\nu=1}^{\mu} \frac{z_\nu^q}{(1-z_\nu)^{2q}}\,\frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)}\, z_\nu^{|k|}, \quad k\in\mathbb{Z},\ \ q\in\mathbb{N}. \tag{4.8}$$

Since $\sum_{k\in\mathbb{Z}} z_\nu^{|k|} = \dfrac{1+z_\nu}{1-z_\nu}$, (4.4) follows.

To establish (4.5), we need some formulae of summation by parts. For $\underline{a} \in \mathfrak{s}(\mathbb{Z})$ and a bounded or polynomially growing sequence $\underline{f}$, it holds

$$\sum_{j\in\mathbb{Z}} f_j D^+ a_j = -\sum_{j\in\mathbb{Z}} (D^- f_j) a_j, \quad \sum_{j\in\mathbb{Z}} f_j D^- a_j = -\sum_{j\in\mathbb{Z}} (D^+ f_j) a_j.$$

For $D = D^+D^-$ these formulae imply

$$\sum_{j\in\mathbb{Z}} f_j D a_j = \sum_{j\in\mathbb{Z}} (D f_j) a_j, \quad \sum_{j\in\mathbb{Z}} f_j D^p a_j = \sum_{j\in\mathbb{Z}} (D^p f_j) a_j, \quad p\in\mathbb{N}. \tag{4.9}$$

Recalling that $\underline{e} = (e_j) = (\delta_{j,0})$, we obtain with the help of (4.3) and (4.9)

$$\begin{aligned} d_k &= \sum_{j\in\mathbb{Z}} a_{k-j} f_j = \sum_{j\in\mathbb{Z}} f_{k-j} a_j = \sum_{j\in\mathbb{Z}} f_{k-j}\Big(\sum_{q=0}^{p-1}\gamma_q D^q \underline{e} + D^p J^p \underline{a}\Big)_j \\ &= \sum_{q=0}^{p-1}\gamma_q \sum_{j\in\mathbb{Z}} (D^q f_{k-j}) e_j + \sum_{j\in\mathbb{Z}} (D^p f_{k-j})(J^p\underline{a})_j \\ &= \sum_{q=0}^{p-1}\gamma_q D^q f_k + \sum_{j\in\mathbb{Z}} (J^p\underline{a})_{k-j} D^p f_j. \end{aligned}$$

We took into account that $D_j f_{k-j} = D_k f_{k-j}$, where the designations $D_j f_{k-j}$ and $D_k f_{k-j}$ mean that the second central difference $D f_{k-j}$ is taken with respect to j or k, respectively; due to the equality of these differences, we may omit the indexes j or k by D. Thus the first expression form (4.5), (4.6) for

d_k is established. Observing that for γ_q and $a_j^{(p)}$ defined in (4.4) and (4.7), it holds

$$f_k + \sum_{q=1}^{p-1} \gamma_q D^q f_k = \sum_{|j|\le p-1} a_j^{(p)} f_{k-j},$$

we obtain also the second representation form (4.5) for d_k. ■

Lemma 2. *Assume that $f \in C(\mathbb{R})$ is bounded or polynomially growing as $|x| \to \infty$. Then for $i \in \mathbb{Z}$, $p \in \mathbb{N}$, there hold the representations*

$$f\Big(\Big(i+\frac{m}{2}\Big)h\Big) - (Q_{h,m}^{(p)} f)\Big(\Big(i+\frac{m}{2}\Big)h\Big) \tag{4.10}$$

$$= \sum_{j=-\mu+1}^{\mu-1} \Big(\sum_{\nu=1}^{\mu} \frac{z_\nu^p}{(1-z_\nu)^{2p}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{|k|\le\mu} b_k\, z_\nu^{|k-j|} \Big) D^p f_{i+j}$$

$$= \sum_{j=-\mu}^{\mu} \Big(\sum_{\nu=1}^{\mu} \frac{z_\nu^p}{(1-z_\nu)^{2p}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{|k|\le\mu} b_k\, D_j z_\nu^{|k-j|} \Big) D^{p-1} f_{i+j}$$

$$= \sum_{j=-\mu+1}^{\mu} \Big(\sum_{\nu=1}^{\mu} \frac{z_\nu^p}{(1-z_\nu)^{2p}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{k=-\mu}^{\mu} b_k (z_\nu^{|k-j|} - z_\nu^{|k-j-1|}) \Big) D^{p-1} D^+ f_{i+j},$$

where $\mu = \mathrm{int}((m-1)/2)$, b_k are defined in (3.4), and z_ν, $\nu = 1, \ldots, \mu$, are the characteristic roots (the roots of the characteristic polynomial $P_{2\mu}$) in $(-1, 0)$ whereas the index j in $D_j z_\nu^{|k-j|}$ indicates that the difference $D = D^+D^-$ is applied to $z_\nu^{|k-j|}$ with respect to j.

Proof. Due to (4.5)–(4.6),

$$(Q_{h,m} f)(x) - (Q_{h,m}^{(p)} f)(x) = \sum_{k\in\mathbb{Z}} \delta_k^{(p)} B_m(h^{-1}x - k) \text{ with } \delta_k^{(p)} = \sum_{j\in\mathbb{Z}} (J^p \underline{a})_{k-j} D^p f_j;$$

due to (3.4) and (4.8),

$$f\left(\left(i+\frac{m}{2}\right)h\right) - (Q_{h,m}^{(p)} f)\left(\left(i+\frac{m}{2}\right)h\right)$$
$$= (Q_{h,m} f)\left(\left(i+\frac{m}{2}\right)h\right) - (Q_{h,m}^{(p)} f)\left(\left(i+\frac{m}{2}\right)h\right)$$
$$= \sum_{k\in\mathbb{Z}} \delta_k^{(p)} B_m\left(i+\frac{m}{2}-k\right) = \sum_{k\in\mathbb{Z}} b_{i-k} \delta_k^{(p)}$$
$$= \sum_{k\in\mathbb{Z}} b_{-k} \delta_{k-i}^{(p)} = \sum_{k\in\mathbb{Z}} b_k \delta_{k-i}^{(p)}$$
$$= \sum_{|k|\le\mu} b_k \Big(\sum_{j\in\mathbb{Z}} (J^p \underline{a})_{k-i-j} D^p f_j \Big) = \sum_{|k|\le\mu} b_k \Big(\sum_{j\in\mathbb{Z}} (J^p \underline{a})_{k-j} D^p f_{i+j} \Big)$$
$$= \sum_{j\in\mathbb{Z}} \Big(\sum_{|k|\le\mu} b_k (J^p \underline{a})_{k-j} \Big) D^p f_{i+j}$$

$$= \sum_{j\in\mathbb{Z}} \Big(\sum_{\nu=1}^{\mu} \frac{z_\nu^p}{(1-z_\nu)^{2p}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{|k|\le\mu} b_k\, z_\nu^{|k-j|} \Big) D^p f_{i+j}.$$

Representing $D^p f_{i+j} = D\, D^{p-1} f_{i+j} = D^- D^{p-1} D^+ f_{i+j}$ and using the summation formulae, in particular,

$$\sum_{j\in\mathbb{Z}} a_j D^- f_j = -\sum_{j\in\mathbb{Z}} (D^+ a_j) f_j = \sum_{j\in\mathbb{Z}} (a_j - a_{j+1}) f_j,$$

we obtain also

$$f\Big(\Big(i+\frac{m}{2}\Big)h\Big) - (Q_{h,m}^{(p)} f)\Big(\Big(i+\frac{m}{2}\Big)h\Big)$$

$$= \sum_{j\in\mathbb{Z}} \Big(\sum_{\nu=1}^{\mu} \frac{z_\nu^p}{(1-z_\nu)^{2p}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{|k|\le\mu} b_k D_j z_\nu^{|k-j|} \Big) D^{p-1} f_{i+j}$$

$$= \sum_{j\in\mathbb{Z}} \Big(\sum_{\nu=1}^{\mu} \frac{z_\nu^p}{(1-z_\nu)^{2p}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{|k|\le\mu} b_k \Big(z_\nu^{|k-j|} - z_\nu^{|k-j-1|} \Big) \Big) D^{p-1} D^+ f_{i+j}.$$

These formulae take the form (4.10) since for the characteristic values z_ν we have $\sum_{|k|\le\mu} b_k\, z_\nu^{|k-j|} = 0$ for $|j| \ge \mu$:

$$\text{for } j \le -\mu, \quad \sum_{|k|\le\mu} b_k\, z_\nu^{|k-j|} = \sum_{|k|\le\mu} b_k\, z_\nu^{k-j} = z_\nu^{-j} \sum_{|k|\le\mu} b_k\, z_\nu^{k} = 0,$$

$$\text{for } j \ge \mu, \quad \sum_{|k|\le\mu} b_k\, z_\nu^{|k-j|} = \sum_{|k|\le\mu} b_k\, z_\nu^{j-k} = z_\nu^{j} \sum_{|k|\le\mu} b_k\, z_\nu^{-k} = 0.$$

Recall that together with z_ν, also z_ν^{-1} is a characteristic value. ■

Theorem 2. *For $f \in V^{2p,\infty}(\mathbb{R})$, it holds*

$$\|Q_{h,m} f - Q_{h,m}^{(p)} f\|_\infty \le q_m \sup_{i\in\mathbb{Z}} \Big| f\Big(\Big(i+\frac{m}{2}\Big)h\Big) - (Q_{h,m}^{(p)} f)\Big(\Big(i+\frac{m}{2}\Big)h\Big) \Big|$$

$$\le q_m c_m^{(p)} h^{2p} \|f^{(2p)}\|_\infty, \tag{4.11}$$

where $q_m = \|Q_{h,m}\|_{BC(\mathbb{R})\to BC(\mathbb{R})}$ and

$$c_m^{(p)} = \sum_{j=-\mu+1}^{\mu-1} \Big| \sum_{\nu=1}^{\mu} \frac{z_\nu^p}{(1-z_\nu)^{2p}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{|k|\le\mu} b_k\, z_\nu^{|k-j|} \Big|.$$

Proof. Clearly, $Q_{h,m} f - Q_{h,m}^{(p)} f = Q_{h,m}(f - Q_{h,m}^{(p)} f)$, and (4.11) follows with the help of the first one of representations (4.10). ■

Differently from $\|f - Q_{h,m} f\|_\infty$ which is saturated at the accuracy $O(h^m)$, there is no saturation in the error $\|Q_{h,m} f - Q_{h,m}^{(p)} f\|_\infty$ – according to (4.11), the accuracy order $O(h^{2p})$ grows with p if f is sufficiently regular. It is reasonable to quasi-interpolate with the smallest $p \in \mathbb{N}$ for which $2p > m$; denote it by p', i.e.,

$$p' = \operatorname{int}\left(\frac{m+2}{2}\right) = \left\{\begin{array}{l} \frac{m}{2} + 1,\ m \text{ even} \\ \frac{m+1}{2},\ m \text{ odd} \end{array}\right\}.$$

Denote also $Q'_{h,m} := Q^{(p')}_{h,m}$, $a'_{j,m} := a^{(p')}_{j,m}$. As we see from Theorem 3 below, $\|f - Q'_{h,m}f\|_\infty$ asymptotically maintains the accuracy of $\|f - Q_{h,m}f\|_\infty$ for C^m-smooth f.

Note that a quasi-interpolant can be determined from local values of f since for $x \in [ih, (i+1)h]$, $i \in \mathbb{Z}$, the sum in (4.1) is restricted to the following terms:

$$(Q^{(p)}_{h,m}f)(x) = \sum_{k=i-m+1}^{i} \Big(\sum_{|j|\le p-1} a^{(p)}_j f_{k-j} \Big) B_m(h^{-1}x - k).$$

In this expression, index $k - j$ varies between $(i - m + 1) - (p - 1)$ and $i + (p - 1)$, and $f_{k-j} = f((k - j + \frac{m}{2})h)$ exploits values of f from the interval $[(i - \frac{m}{2} - p + 2)h, (i + \frac{m}{2} + p - 1)h]$. Thus $(Q^{(p)}_{h,m}f)(x)$ is well defined for $x \in [ih, (i+1)h]$ if f is given on the interval $[(i - \frac{m}{2} - p + 2)h, (i + \frac{m}{2} + p - 1)h]$. Also the total error $f(x) - (Q^{(p)}_{h,m}f)(x)$ can be estimated locally for any $p \in \mathbb{N}$. We restrict ourselves to the case $p = p'$ and $x \in [0, 1]$. The quasi-interpolant

$$(Q'_{h,m}f)(x) = \sum_{k=-m+1}^{n-1} \Big(\sum_{|j|\le p'-1} a'_{j,m} f_{k-j} \Big) B_m(h^{-1}x - k),\ \ 0 \le x \le 1,$$

is well defined for $f \in C(-mh, 1 + mh)$.

Theorem 3. *For $m \ge 3$, $f \in W^{m,\infty}(-\delta, 1 + \delta)$, $\delta \ge mh$, it holds*

$$\max_{0\le x\le 1} |f(x) - (Q'_{h,m}f)(x)| \le (\Phi_{m+1}\pi^{-m} + q_m c'_m)h^m \sup_{-\delta<x<1+\delta} |f^{(m)}(x)|, \tag{4.12}$$

where for even m,

$$c'_m = \sum_{j=-\mu}^{\mu} \left| \sum_{\nu=1}^{\mu} \frac{z_\nu^{p'}}{(1 - z_\nu)^{2p'}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{k=-\mu}^{\mu} b_k D_j z_\nu^{|k-j|} \right|,$$

whereas for odd m,

$$c'_m = \sum_{j=-\mu+1}^{\mu} \left| \sum_{\nu=1}^{\mu} \frac{z_\nu^{p'}}{(1 - z_\nu)^{2p'}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{k=-\mu}^{\mu} b_k (z_\nu^{|k-j|} - z_\nu^{|k-j-1|}) \right|.$$

Moreover, for any compact subset M of $C^m[-\delta, 1 + \delta]$, it holds

$$\sup_{f\in M}\max_{0\le x\le 1}|f(x)-(Q'_{h,m}f)(x)| \le \Phi_{m+1}\pi^{-m}h^m \max_{-\delta\le x\le 1+\delta}|f^{(m)}(x)|+h^m\varepsilon_{h,m,M}, \quad (4.13)$$

where $\varepsilon_{h,m,M}\to 0$ *as* $h\to 0$.

Proof. Let us extend $f\in W^{m,\infty}(-\delta,1+\delta)\subset C^{m-1}[-\delta,1+\delta]$ up to function $\overline{f}\in V^{m,\infty}(\mathbb{R})$ by setting

$$\overline{f}(x)=\left\{\begin{array}{cc} f_-(x), & x<-\delta \\ f(x), & -\delta\le x\le 1+\delta \\ f_+(x), & x>1+\delta\end{array}\right\}$$

where $f_\mp$ are the Taylor polynomials of f of degree $m-1$ with expansion centers $-\delta$ and $1+\delta$, respectively. For $0\le x\le 1$ we have

$$f(x)-(Q'_{h,m}f)(x)=\overline{f}(x)-(Q'_{h,m}\overline{f})(x),$$

and together with the equality

$$\overline{f}-Q'_{h,m}\overline{f}=\overline{f}-Q_{h,m}\overline{f}+Q_{h,m}(Q_{h,m}\overline{f}-Q'_{h,m}\overline{f})$$

we obtain

$$\max_{0\le x\le 1}|f(x)-(Q'_{h,m}f)(x)| \le \|\overline{f}-Q_{h,m}\overline{f}\|_\infty+q_m\sup_{l\in\mathbb{Z}}\left|\overline{f}\left(\left(l+\frac{m}{2}\right)h\right)-(Q^{(p')}_{h,m}\overline{f})\left(\left(l+\frac{m}{2}\right)h\right)\right|.$$

By Theorem 1,

$$\|\overline{f}-Q_{h,m}\overline{f}\|_\infty \le \Phi_{m+1}\pi^{-m}h^m\|\overline{f}^{(m)}\|_\infty.$$

Using the second and third representation (4.10) respectively for even and odd m, we get

$$\sup_{l\in\mathbb{Z}}\left|\overline{f}\left(\left(l+\frac{m}{2}\right)h\right)-(Q^{(p')}_{h,m}\overline{f})\left(\left(l+\frac{m}{2}\right)h\right)\right| \le c'_m\left\{\begin{array}{c}\sup_{j\in\mathbb{Z}}|D^{p'-1}\overline{f}_j|,\ m \text{ even} \\ \sup_{j\in\mathbb{Z}}|D^{p'-1}D^+\overline{f}_j|,\ m \text{ odd}\end{array}\right\}.$$

Further, for even and odd m we have, respectively,

$$|D^{p'-1}\overline{f}_j| = |D^{m/2}\overline{f}_j| \le h^m\|\overline{f}^{(m)}\|_\infty = h^m\|f^{(m)}\|_\infty,$$

$$|D^{p'-1}D^+\overline{f}_j| = |(D^{(m-1)/2}D^+)\overline{f}_j| \le h^m\|\overline{f}^{(m)}\|_\infty = h^m\|f^{(m)}\|_\infty,$$

where $\|f^{(m)}\|_\infty := \sup_{-\delta<x<1+\delta}|f^{(m)}(x)|$. Thus

$$\|Q_{h,m}(Q_{h,m}\overline{f} - Q_{h,m}^{(p')}\overline{f})\|_\infty \le q_m c'_m h^m \sup_{-\delta<x<1+\delta} |f^{(m)}(x)| \tag{4.14}$$

and (4.12) follows.

To prove (4.13), introduce the operator

$$A_{h,m} : C^m[-\delta, 1+\delta] \to C[0,1], \;\; A_{h,m}f = h^{-m}Q_{h,m}(Q_{h,m}\overline{f} - Q'_{h,m}\overline{f}),$$

where the extension $\overline{f}$ of f now is built using the Taylor polynomials of f of degree m. For $f \in C^{m+1}[-\delta, 1+\delta]$, we then have $\overline{f} \in V^{m+1,\infty}(\mathbb{R})$, and similarly as (4.14) we obtain (cf. also (4.11)) an estimate of order

$$\|Q_{h,m}(Q_{h,m}\overline{f} - Q_{h,m}^{(p')}\overline{f})\|_\infty = O(h^{m+1}).$$

Thus $\|A_{h,m}f\|_{C[0,1]} \to 0$ as $h \to 0$ for f from $C^{m+1}[-\delta, 1+\delta]$ which is a dense set in $C^m[-\delta, 1+\delta]$. According to (4.14), $\|A_{h,m}\|_{C^m[-\delta,1+\delta]\to C[0,1]} \le q_m c'_m$ for all sufficiently small h (for $h \le \delta/m$). By Banach–Steinhaus theorem, the convergence $\|A_{h,m}f\|_{C[0,1]} \to 0$ as $h \to 0$ takes place for all $f \in C^m[-\delta, 1+\delta]$; the convergence is uniform with respect to $f \in M$ where $M \subset C^m[-\delta, 1+\delta]$ is a compact set. This proves (4.13) with $\varepsilon_{h,m,M} = \sup_{f\in M} \|A_{h,m}f\|_{C[0,1]} \to 0$ as $h \to 0$. ■

The weights $a'_{j,m} := a_{j,m}^{(p')} = a_{-j,m}^{(p')} = a'_{-j,m}$, $j = 0, \ldots, p'-1$, of the quasi-interpolant

$$(Q'_{h,m}f)(x) = \sum_{k\in\mathbb{Z}} \Big(\sum_{|j|\le p'-1} a'_{j,m} f\Big(\Big(k - j + \frac{m}{2}\Big)h\Big)\Big) B_m(h^{-1}x - k)$$

can be computed by (4.7) once for ever. For $m = 3, \ldots, 10$ they are as follows:

Table 2. The weights $a'_{j,m}$ of the quasi-interpolant.

m	$a'_{0,m}$	$a'_{1,m}$	$a'_{2,m}$	$a'_{3,m}$	$a'_{4,m}$	$a'_{5,m}$
3	1.2500000	-0.1250000				
4	1.5000000	-0.2777778	0.0277778			
5	1.6614583	-0.3715278	0.0407986			
6	2.0541667	-0.6385417	0.1229167	-0.0114583		
7	2.3113137	-0.8030165	0.1629774	-0.0156178		
8	2.9285825	-1.2534083	0.3430732	-0.0587258	0.0047696	
9	3.3532232	-1.5474118	0.4418932	-0.0774754	0.0063823	
10	4.3468295	-2.3113639	0.8030947	-0.1918579	0.0287522	-0.0020398

The values of $q'_m := \|Q'_{h,m}\|_{BC(\mathbb{R})\to BC(\mathbb{R})}$ can be computed according to the formula (cf. the formula for q_m in Section 3)

$$q_m^{(p)} := \|Q_{h,m}^{(p)}\|_{BC(\mathbb{R})\to BC(\mathbb{R})} = \max_{x\in[\frac{m}{2},\frac{m+1}{2}]} \sum_{j\in\mathbb{Z}} |F_{m,p}(x+j)|,$$

$$F_{m,p}(x) := \sum_{|k|\le p-1} a_{k,m}^{(p)} B_m(x-k), \quad p \in \mathbb{N}.$$

For $m = 3, \ldots, 10$, the numerical values of c'_m (see Theorem 3) and q'_m can be found in Table 3.

Table 3. Numerical values of c'_m and q'_m.

m	3	4	5	6	7	8	9	10	20
c'_m	0.016	0.019	0.015	0.0085	0.0060	0.0030	0.0022	0.0010	$6.5 \cdot 10^{-6}$
q'_m	1.250	1.354	1.329	1.403	1.356	1.413	1.378	1.419	1.514
q_m	1.414	1.549	1.706	1.816	1.916	2.000	2.075	2.142	2.583

For a comparison, we recalled also the values of $q_m = \|Q_{h,m}\|_{BC(\mathbb{R})\to BC(\mathbb{R})}$. We see that the quasi-interpolation process is even more stable than the interpolation process. On the basis of presented numerical values, it is difficult to set a hypothesis whether q'_m is bounded or of a logarithmic growth as $m \to \infty$; the latter hypothesis seems to be more probable.

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References

[1] C. de Boor. *A practical Guide to Splines.* Springer, New York, 2001.
[2] N. P. Korneichuk. *Splines in the Approximation Theory.* Nauka, Moscow, 1984. (In Russian)
[3] A. Pedas and G. Vainikko. Smoothing transformation and piecewise polynomial projection methods for weakly singular Fredholm integral equations. *Comm. Pure Appl. Anal.*, **5**, 395–413, 2006.
[4] L. L. Schumaker. *Spline Functions: Basic Theory.* Krieger Publ., Malabar, Florida, 1993.
[5] S. B. Stechkin and Yu. N. Subbotin. *Splines in Numerical Mathematics.* Nauka, Moscow, 1976. (In Russian)
[6] G. Vainikko. *Error estimates of spline interpolation on the real line.* (In progress)
[7] Yu. S. Zav'yalov, B. I. Kvasov and V. L. Miroshnichenko. *Methods of Spline Functions.* Nauka, Moscow, 1980. (In Russian)