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GRID APPROXIMATION OF SINGULARLY PERTURBED PARABOLIC EQUATIONS WITH MOVING BOUNDARY LAYERS *

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Dedicated to the memory of Academician Alexandr Andreevich Samarskii

Abstract. A grid approximation of a boundary value problem is considered for a singularly perturbed parabolic reaction–diffusion equation in a domain with boundaries moving along the x-axis in the positive direction. For small values of the parameter ε (that is the coefficient of the highest-order derivative in the equation, $\varepsilon \in (0,1]$), a *moving boundary layer* appears in a neighbourhood of the left lateral boundary S_1^L. It turns out that, in the class of difference schemes on rectangular grids condensing in a neighbourhood of S_1^L with respect to x and t, there do not exist schemes that converge even under the condition $P_0^{-1} \approx \varepsilon^{1/2}$, where P_0 is the total number of nodes in the meshes used, that is, $P_0 \approx NN_0$, where the values N and N_0 define the numbers of mesh points in x and t. On such meshes, *convergence under the condition* $N^{-1} + N_0^{-1} \le \varepsilon^{1/4}$ *cannot be achieved.* Examination of widths similar to Kolmogorov's widths allows us to establish necessary and sufficient conditions for the ε-uniform convergence of approximations to the solution of the boundary value problem. Using these conditions, a scheme is constructed on a mesh being piecewise uniform in a coordinate system adapted to the moving boundary. This scheme *converges ε-uniformly* at the rate $\mathcal{O}(N^{-1}\ln N + N_0^{-1})$.

Key words: boundary value problem, perturbation parameter ε, parabolic reaction–diffusion equation, finite difference approximation, moving boundary layer, Kolmogorov's widths, ε-uniform convergence.

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1 Introduction

For singularly perturbed boundary value problems, the development of special grid methods for which the error in the solutions depends weakly on the parameter ε, in particular, ε-uniformly convergent methods is known to be difficult. At present, a method for constructing ε-uniformly convergent schemes on special meshes that are *a priori* condensing in boundary layers is well developed (see, e.g., [2, 3, 11, 13, 18] in the case of partial differential equations). Methods based on piecewise uniform meshes that condense in boundary layers are used fairly widely (see, e.g., [3, 11, 13, 18] and the references therein).

Note that special numerical methods for singularly perturbed parabolic equations have been studied intensively only for problems with stationary boundary and interior layers. There are some specific features in constructing special difference schemes in the case of moving boundary and interior layers. For the pure initial-value singularly perturbed problem with a moving concentrated source in a neighbourhood of which a moving interior layer appears, special ε-uniformly convergent difference schemes were constructed in [20, 21, 25]. To construct such schemes, nonrectangular grids that condense along the x-axis were used in a neighbourhood of the trajectory of the moving source. Such schemes are fairly complicated, that draws our attention to methods for constructing simpler schemes and alternative numerical methods (based on *a posteriori* adapted grids) that converge ε-uniformly.

In the monograph [14] (see also [15]), A.A. Samarskii have been pointed out that, when constructing numerical methods for regular boundary value problems with sufficiently complicated solutions, in order to prevent appearance of nonphysical effects in the computed solutions, it is necessary to use discrete approximations that inherit monotonicity of the boundary value problem. For singularly perturbed boundary value problems, such natural requirement bring to quite complicated finite difference schemes (in the presence of mixed derivatives in equations, see, e.g., schemes in [16] for regular problems, and in [18] for singularly perturbed problems). Therefore, it is important to impose conditions on the schemes under the construction, which are sufficient and close to necessary one, that guarantee monotonicity and ε-uniform convergence of these schemes.

In the case of singularly perturbed problems with moving boundary and interior layers, investigation of necessary and sufficient conditions for the ε-uniform convergence of numerical methods is very important.

In this paper, we consider the boundary value problem for a singularly perturbed parabolic reaction-diffusion equation in a domain whose boundaries move in the positive direction of the x-axis. For small values of the parameter ε, a moving boundary layer appears in a neighbourhood of the left lateral boundary S_1^L. The derivatives of the solution with respect to x and t grow unboundedly in a neighbourhood of the boundary layer as $\varepsilon \to 0$.

Note that for problems of this type, classical finite difference schemes based on uniform grids converge only when $N^{-1} + N_0^{-1} \ll \varepsilon$, where N and N_0 define the number of mesh points in x and t (see Theorem 2 in Section 4). It turns out that, in the class of difference schemes based on rectangular grids that condense in a neighbourhood of S_1^L with respect to x and t, there are *no convergent schemes even under the condition* $P_0^{-1} \approx \varepsilon^{1/2}$, where P_0 is the number of nodes in the meshes used and $P_0 \approx N\,N_0$ (see Remark 4 of Theorem 4 in Section 5).

A consideration of widths that are similar to Kolmogorov's widths makes it possible to find necessary and sufficient conditions for the ε-uniform convergence of a difference scheme on grids that are not the tensor product of meshes with respect to x and t. These conditions are used to construct a scheme that converges ε-uniformly in the maximum norm. In Section 7, some remarks and generalizations are given related to applications of *the technique based on the widths* for an investigation of ε-uniformly convergent difference schemes.

Some aspects of the construction and investigation of ε-uniformly convergent difference schemes in the maximum norm for the problem under consideration by means of widths are discussed in [24]. A difference scheme and numerical experiments for a problem with moving boundaries are given in [12].

When studying ε-uniformly convergent numerical methods, the potential usefulness of widths was already mentioned by N.S. Bakhvalov in the 1970s. An application of widths to the study of optimal L_2-convergence rates for approximations of solutions to singularly perturbed elliptic problems is given in [6], [7]; see also [10] and the bibliographies of these papers. But the widths in spaces with L_2-norm do not allow to investigate ε-uniform convergence of numerical methods in the maximum norm.

2 Problem Formulation. The Aim of the Study

2.1. In the domain $\overline{G}$ with the boundary $S = \overline{G} \setminus G$, where

$$G = \{(x,t):\ \beta_1(t) < x < \beta_2(t),\quad t \in (0,T]\,\}, \tag{2.1}$$

consider the boundary value problem for the singularly perturbed parabolic equation

$$\begin{aligned} &L\,u(x,t) \equiv \left\{ \varepsilon\,\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \right\} u(x,t) = f(x,t), \quad (x,t) \in G, \qquad (2.2)\\ &u(x,t) = \varphi(x,t), \quad (x,t) \in S. \end{aligned}$$

Here $f(x,t)$, $(x,t) \in \overline{G}$, $\varphi(x,t)$, $(x,t) \in S$, and $\beta_i(t)$, $t \in [0,T]$, $i = 1,2$, are sufficiently smooth functions satisfying the conditions

$$\begin{aligned} &|f(x,t)| \le M, \quad (x,t) \in \overline{G}, \quad |\,\varphi(x,t)\,| \le M, \quad (x,t) \in S; \qquad (2.3)\\ &0 < v_0 \le (\,d/dt\,)\beta_i(t) \equiv v_i(t) \le v^0, \quad m_1 \le \beta_2(t) - \beta_1(t) \le M_1,\\ &t \in [0,T],\ i = 1,2, \end{aligned}$$

moreover, $\beta_1(0) = 0$, $\beta_2(0) = d$; the parameter ε takes arbitrary values in the open-closed interval $(0,1]$, and the derivatives $\beta_i'(t)$ specify the velocity of the moving lateral boundaries. We assume that the boundary S consists of the sets S^L and S_0, that is, $S = S_0 \cup S^L$, where S^L is the lateral boundary;

$S^L = S_1^L \cup S_2^L$, where S_1^L and S_2^L are, respectively, the left and the right boundaries of the set G, S_0 is the lower base of G with $S_0 = \overline{S}_0$.

We assume that compatibility conditions ensure the sufficient smoothness of the solution for fixed ε (see [8]) on the set $S^c = S_0 \cap \overline{S}^L$, i.e., at the corner points $(0,0)$ and $(d,0)$.

As $\varepsilon \to 0$, a moving boundary layer appears in a neighbourhood of the set S_1^L. This layer exponentially decreases when moving away from S_1^L as x increases and/or t decreases (see estimate (3.5) in Section 3).

2.2. Unlike regular problems, in the case of singularly perturbed problems the ε-uniform convergence of the grid solution $z(x,t)$ at the nodes of the mesh $\overline{G}_h$ in the maximum discrete norm is, in general, inadequate to describe the ε-uniform convergence of the approximation constructed on the set $\overline{G}$. The convergence of the grid solution on the mesh $\overline{G}_h$ does not imply the convergence of its interpolants on the set $\overline{G}$.

We give some definitions. In the case when the interpolant $\overline{z}_{(4.5)}(x,t)$, $(x,t) \in \overline{G}$, converges on $\overline{G}$, we say that the *difference scheme resolves the boundary value problem* (for some values of the parameter ε); otherwise, we say that the difference scheme does not resolve the boundary value problem. When the interpolant $\overline{z}_{(4.5)}(x,t)$, $(x,t) \in \overline{G}$, converges on $\overline{G}$ ε-uniformly, we say that the difference scheme resolves the boundary value problem *ε-uniformly*.

Let $\overline{G}_h$ be some grid, and let $\overline{u}^h(x,t)$, $(x,t) \in \overline{G}$, be a linear interpolant constructed using the solution $u(x,t)$ of the boundary value problem at the nodes of the mesh $\overline{G}_h$. If some such interpolant $\overline{u}^h(x,t)$ converges on $\overline{G}$ as N and $N_0 \to \infty$ (for some values of the parameter ε), we say that the *mesh $\overline{G}_h$ is informative* (and if $\overline{u}^h(x,t)$ converges ε-uniformly then the mesh $\overline{G}_h$ is ε-uniformly informative); otherwise, we say that the mesh $\overline{G}_h$ is not informative. Here N and N_0 define the number of mesh points in x and t, respectively.

The informativity of the grid $\overline{G}_h$ is a necessary condition for the boundary value problem (2.2), (2.1) to be resolved by the difference scheme on $\overline{G}_h$.

We say that the solution of a difference scheme (or, briefly, the scheme itself) converges if the grid solution converges on $\overline{G}_h$ and the difference scheme resolves the boundary value problem. But if the solution converges only on the mesh $\overline{G}_h$, however, the solvability of the boundary value problem is, in general, not assumed, we say that the *solution* (or, the scheme) *converges on the mesh* $\overline{G}_h$.

2.3. Errors in the solutions of difference schemes based on the classical approximations of problem (2.2), (2.1) depend on the parameter ε and become small only when ε is essentially greater than, e.g., the "maximum" step-sizes of the meshes in x and t. So, by virtue of bounds (4.8) and (4.11), the classical difference scheme (4.4), (4.7) (see Section 4) converges under the condition $N^{-1} + N_0^{-1} \ll \varepsilon$, or more precisely,

$$\varepsilon^{-1} = o\big(\min[N,\, N_0]\big), \quad N,\, N_0 \to \infty. \tag{2.4}$$

If this condition is violated, the solution of the difference scheme does not converge to the solution of the problem (2.2), (2.1). Condition (2.4) is more restrictive than the condition $N^{-1} \ll \varepsilon$, or more precisely,

$$\varepsilon^{-1} = o\big(N\big), \quad N,\, N_0 \to \infty, \tag{2.5}$$

i.e., the convergence condition of the classical scheme for problems in domains with fixed boundaries. Note that there are no constraints on the t-mesh-size in the condition (2.5).

Thus, because of the above behaviour of the grid solutions that approximate the solution of the differential problem with a moving boundary layer, it is necessary to construct special difference schemes in which the solution error is independent of ε. In particular, it is of interest to have schemes that converge under a weaker condition than (2.4), which is a condition for the convergence of solutions of the scheme (4.4), (4.7).

The conditions imposed on the grid approximations of problem (2.2), (2.1) that are necessary and sufficient for ε-uniform (or close to it) convergence of grid solutions are of great importance.

Further, we need some definitions in the case of difference schemes on meshes with an arbitrary distribution of mesh points.

Definitions. Let $E_\varepsilon = \{\varepsilon : \ \varepsilon \in (0,1]\}$. Let $E_{\overline{N}}$ be a subset of the set of pairs of positive integers $(N,\ N_0)$ satisfying the condition $N,\, N_0 \ge M_0$. Let the functions $\psi_i(N^{-1}, N_0^{-1}, \varepsilon)$ for $i = 1, 2$ be defined on the set $E_{\overline{N},\,\varepsilon} = E_{\overline{N}} \times E_\varepsilon$ and satisfy $\psi_i(N^{-1}, N_0^{-1}, \varepsilon) > 0$. The notation

$$\psi_1(N^{-1}, N_0^{-1}, \varepsilon)\, \widehat{o}\left(\psi_2(N^{-1}, N_0^{-1}, \varepsilon)\right) \quad \text{on} \quad E_{\overline{N},\,\varepsilon}$$

means that one can find a point $\left(\widetilde{N}_1^{-1}, \widetilde{N}_2^{-1}, \widetilde{\varepsilon}\right)$ such that the relation

$$\psi_1(N^{-1}, N_0^{-1}, \varepsilon)\left[\psi_2(N^{-1}, N_0^{-1}, \varepsilon)\right]^{-1} \to 0$$

as $(N^{-1}, N_0^{-1}, \varepsilon) \to (\widetilde{N}^{-1}, \widetilde{N}_0^{-1}, \widetilde{\varepsilon})$, $(N^{-1}, N_0^{-1}, \varepsilon)$, $(\widetilde{N}^{-1}, \widetilde{N}_0^{-1}, \widetilde{\varepsilon}) \in E_{\overline{N},\,\varepsilon}$, is fulfilled. For a grid function $z(x,\, t)$, $(x,\, t) \in \overline{G}_h$, i.e., a solution of a difference scheme, assume that the estimate

$$|u(x,\, t) - z(x,\, t)| \le M\, \mu(N^{-1}, N_0^{-1}, \varepsilon), \qquad (x,\, t) \in \overline{G}_h$$

holds. Here the grid $\overline{G}_h$ is a tensor product of meshes in x and t where $N+1$ and N_0+1 are the number of nodes in x and t respectively. We say that this *estimate is unimprovable* with respect to the values N, N_0, ε if the estimate

$$|u(x,\, t) - z(x,\, t)| \le M\, \mu_0(N^{-1}, N_0^{-1}, \varepsilon), \quad (x,\, t) \in \overline{G}_h,$$

is, in general, not valid if $\mu_0(N^{-1}, N_0^{-1}, \varepsilon) = \widehat{o}(\mu(N^{-1}, N_0^{-1}, \varepsilon))$ on $E_{\overline{N},\,\varepsilon}$.

Assume that a solution of a difference scheme converges to the solution of the boundary value problem as $N,\, N_0 \to \infty$ and $\varepsilon \in E_\varepsilon$ when $N^{-1},\, N_0^{-1} = o\big(\varepsilon^\nu\big)$ and $\varepsilon \in E_\varepsilon$, but convergence under the condition $N^{-1},\, N_0^{-1} = O(\varepsilon^\nu)$ does not, in general, occur. In this case we say that the difference scheme *converges with defect* ν with respect to the parameter ε as $N,\, N_0 \to \infty$ (or, briefly, the scheme converges with defect ν). When $\nu = 0$, the convergence is ε-uniform.

Let $\overline{G}_h$ be a grid (in general, not a rectangular one) on the set $\overline{G}$, and let P_0 be the number of mesh points in $\overline{G}_h$. Let the grid function $z(x,t)$, $(x,t) \in \overline{G}_h$, be a solution of some difference scheme that converges with the bound

$$|u(x,t) - z(x,t)| \le M\,\mu\big(P_0^{-1},\, \varepsilon\big), \quad (x,t) \in \overline{G}_h. \tag{2.6}$$

Unimprovability of an estimate with respect to the values P_0, ε is defined similarly to the definition with respect to the values N, N_0, ε.

If the solution of a difference scheme converges to the solution of the boundary value problem as $P_0 \to \infty$, $\varepsilon \in E_\varepsilon$, under the condition $P_0^{-1/2} = o(\varepsilon^\nu)$, $\varepsilon \in E_\varepsilon$, but does not, in general, converge under the condition $P_0^{-1/2} = O(\varepsilon^\nu)$, then the scheme is said to be convergent *with defect* ν (as $P_0 \to \infty$). Note that, as $N \approx N_0$, we have $N^{-1}, N_0^{-1} \approx P_0^{-1/2}$, which motivates the given definition of convergence with defect ν for the scheme on a mesh with P_0 points.

If the value ν can be chosen arbitrarily small, we say that the difference scheme, which is controlled by the value ν, converges *almost ε-uniformly with defect ν* (or, briefly, *almost ε-uniformly*).

Thus, the defect of the ε-uniform convergence of the classical scheme (4.4), (4.7) (with respect to N, N_0, and ε) is equal to unity.

Our aim for the boundary value problem (2.2), (2.1) is to find necessary conditions for the ε-uniform and almost ε-uniform convergence of solutions of difference schemes constructed using classical approximations of the differential equation and, in addition, to construct a difference scheme that converges ε-uniformly.

3 A Priori Estimates

Let us give *a priori* estimates on the solution of the boundary value problem (2.2), (2.1) that are used in the following constructions. On the set $\overline{G}$, we represent the solution as the sum of its regular and singular components:

$$u(x,t) = U(x,t) + W(x,t), \quad (x,t) \in \overline{G}. \tag{3.1}$$

In the problem (2.2), (2.1), we pass to the variables ξ, t in which the lateral boundaries are fixed. It is convenient to transform the variable x into the variable $\xi = \xi(x,t)$ defined by

$$\xi = \xi(x,t) = d\,\big(x - \beta_1(t)\big)\,\big(\beta_2(t) - \beta_1(t)\big)^{-1}, \quad (x,t) \in \overline{G}. \tag{3.2a}$$

We denote the inverse mapping of $\xi(x,t)$ by $\xi^{-1}(\xi,t) \equiv x(\xi,t)$. For the functions $v(x,t)$ and $Z(\xi,t)$ and the subdomains $G^0 \subseteq \overline{G}$, we will use the notation

$$v\big(x(\xi,t),t\big) = v_\xi(\xi,t) = \big\{v(x,t)\big\}_\xi = \widetilde{v}(\xi,t), \tag{3.2b}$$

$$Z\big(\xi(x,t),t\big) = Z_{\xi^{-1}}(x,t) = \big\{Z(\xi,t)\big\}_{\xi^{-1}},$$

$$G^0_\xi = \big\{G^0\big\}_\xi = \xi\big(G^0\big) = \big\{(\xi,t) :\ \big(x(\xi,t),t\big) \in G^0\big\}. \tag{3.2c}$$

We define

$$\widetilde{G}^0_{\xi^{-1}} = \big\{\widetilde{G}^0\big\}_{\xi^{-1}} = \xi^{-1}\big(\widetilde{G}^0\big) = \big\{(x,t) :\ \big(\xi(x,t),t\big) \in \widetilde{G}^0\big\},$$

where $\widetilde{G}^0$ is some subset of a set $\overline{\widetilde{G}}$, and $\overline{\widetilde{G}} = \overline{G}_\xi = \big\{\overline{G}\big\}_\xi$.

In the variables ξ, t, the problem (2.2), (2.1) is transformed into the boundary value problem

$$\begin{cases} \widetilde{L}\,\widetilde{u}(\xi,t) \equiv \left\{ \varepsilon A(\xi,t)\dfrac{\partial^2}{\partial \xi^2} + B(\xi,t)\dfrac{\partial}{\partial \xi} - \dfrac{\partial}{\partial t} \right\} \widetilde{u}(\xi,t) = \widetilde{f}(\xi,t), \quad (\xi,t) \in \widetilde{G}, \\ \widetilde{u}(\xi,t) = \widetilde{\varphi}(\xi,t), \quad (\xi,t) \in \widetilde{S}. \end{cases}$$

Here

$$A(\xi,\, t) = \left\{ \left[\frac{\partial}{\partial x}\xi(x,t) \right]^2 \right\}_{\xi}, \quad B(\xi,\, t) = -\left\{ \left[\frac{\partial}{\partial t}\xi(x,t) \right] \right\}_{\xi}, \quad (\xi,\, t) \in \overline{\widetilde{G}}.$$

Owing to condition (2.3), we have

$$B(\xi,\, t) \geq B_0 > 0, \quad (\xi,\, t) \in \overline{\widetilde{G}}.$$

On the set

$$\overline{\widetilde{G}} = \widetilde{G} \bigcup \widetilde{S}, \quad \widetilde{G} = \widetilde{D} \times (0,T], \quad \widetilde{D} = \{\xi :\ 0 < \xi < d\}, \tag{3.3}$$

problem (3.3) is a boundary value problem for a singularly perturbed parabolic convection-diffusion equation in a domain with a fixed lateral boundary. The boundary layer appears in a neighbourhood of the left side $\widetilde{S}_1^L$ of the lateral boundary $\widetilde{S}^L$, towards which the convective flow is directed.

Let us estimate the regular and singular component in the variables ξ, t (see, e.g., [4, 5]). Returning to the variables x, t, we obtain the estimates

$$\left| \frac{\partial^{k_1+k_0}}{\partial x^{k_1}\partial t^{k_0}}\, U(x,t) \right| \ \leq\ M, \quad (x,t) \in \overline{G}; \quad k_1 + 2k_0 \leq 4, \tag{3.4}$$

$$\left| \frac{\partial^{k_1+k_0}}{\partial x^{k_1}\partial t^{k_0}}\, W(x,t) \right| \ \leq\ M\,\varepsilon^{-k_1-k_0}\, \exp\left(-m_1\,\varepsilon^{-1}\,(x - \beta_1(t)) \right),$$

where m_1 is an arbitrary number in the interval $(0,\, m_0)$, and

$$m_0 = \frac{m_{1(2.3)}}{M_{1(2.3)}} \min_{[0,\,T]} \Big[(d/dt)\beta_1(t) \Big].$$

For the function $W(x,t)$, we also have the estimate

$$\left| \frac{\partial^{k_1+k_0}}{\partial x^{k_1}\partial t^{k_0}}\, W(x,t) \right| \ \leq\ M\,\varepsilon^{-k_1-k_0}\, \exp\left(-m\,\varepsilon^{-1}\, r\big((x,t),\, S_1^L\big) \right), \tag{3.5a}$$
$$(x,t) \in \overline{G}; \quad k_1 + 2k_0 \leq 4,$$

where $r\big((x,t),\, S_1^L\big)$ is the distance from the point (x,t) to the set S_1^L, and m is an arbitrary number in the interval $(0, m^0)$, where

$$m^0 = v_0\big(1 + (v^0)^2\big)^{-1/2}, \quad v_0 = v_{0(2.3)}, \quad v^0 = v^0_{(2.3)}. \tag{3.5b}$$

Thus, unlike problems in domains with fixed boundaries, the derivatives of the singular component $W(x,t)$ with respect to both variables x and t grow unboundedly in a neighbourhood of the moving boundary layer as $\varepsilon \to 0$.

When deriving the estimates, for simplicity we assume that the following compatibility condition on the set S^c (see [8]) is fulfilled:

$$\frac{\partial^{k_1+k_0}}{\partial x^{k_1}\partial t^{k_0}}\,\varphi(x,t)=0,\quad \frac{\partial^{k_1+k_0}}{\partial x^{k_1}\partial t^{k_0}}\,f(x,t)=0,\quad (x,t)\in S^c,\quad k_1+k_0\le l,\quad (3.6)$$

where $l \ge 6$.

Theorem 1. *Let the data of the boundary value problem* (2.2), (2.1) *satisfy the conditions* $f \in C^{\,l+\alpha}(\overline{G})$, $\varphi \in C^{\,l+\alpha}(\overline{G})$, $\beta_i \in C^{\,l+\alpha}([0,T])$, $l = K+4$, $K \ge 2$, $\alpha \in (0,1)$, *and let conditions* (2.3) *and* (3.6) *be fulfilled. Then the components in the representation* (3.1) *for the solution of the boundary value problem* (2.2), (2.1) *satisfy the estimates* (3.4) *and* (3.5).

4 Classical Difference Schemes

Let us write down the classical difference scheme for problem (2.2), (2.1) and discuss some difficulties that arise in the numerical solution when ε is small.

On the strip $\overline{G}^{\infty} = I\!R \times [0,\,T]$ define rectangular *base* grids that will be used to construct the required grids. Let

$$\overline{G}_h = \overline{G}_h^{\,b} = \omega_1 \times \overline{\omega}_0, \qquad (4.1)$$

where ω_1 and $\overline{\omega}_0$ are meshes on the x-axis and on the interval $[0,T]$, respectively; ω_1 and $\overline{\omega}_0$ are meshes with an arbitrary distribution of nodes satisfying only the condition $h \le MN^{-1}$, $h_t \le MN_0^{-1}$, where $h = \max_i\, h^i$, $h^i = x^{i+1}-x^i$, x^i, $x^{i+1}\in\omega_1$, and $h_t = \max_j\, h_t^j$, $h_t^j = t^{j+1}-t^j$, t^j, $t^{j+1}\in\overline{\omega}_0$. Here $N+1$ and N_0+1 are the maximal number of nodes on per unit length on the x-axis and the number of nodes in the mesh $\overline{\omega}_0$, respectively. The grids that are uniform with respect to x and t are of particular interest for us

$$\overline{G}_h = \overline{G}_h^{\,b} = \overline{G}_h^{\,bu}, \qquad (4.2)$$

here grids $\overline{G}_h^{\,bu}$ are $\overline{G}^{\,b}_{h(4.1)}$, where ω_1 and $\overline{\omega}_0$ are uniform meshes with the step-sizes $h = N^{-1}$ and $h_t = TN_0^{-1}$.

On the set $\overline{G}$, we construct the grid (generated by the grid $\overline{G}_{h(4.1)}$)

$$\overline{G}_h = \overline{G}_h(\overline{G}^{\,b}_{h(4.1)}) = G_h \bigcup S_h. \qquad (4.3)$$

The set G_h is the set of nodes (x^i,t^j) in $G\cap\overline{G}_h^{\,b}$ for which the segment $x^i\times(t^{j-1},\,t^j]$ entirely belongs to G. The intersections of the lines $t=t^j$, $t^j\in\omega_0$ with the sides S^L is denoted by S_h^L. The set S_h is formed by S_h^L and the nodes (x^i,t^0) belonging to S_0 for which the segment $x^i\times[t^0,\,t^1)$ entirely belongs to $G\cup S_0$ (this set is denoted by S_{0h}); the nodes $(0,0)$ and $(d,0)$ are assumed to lie in S_{0h}. We set $S_h = S_{0h}\cup S_h^L$.

Problem (2.2), (2.1) is approximated by the implicit difference scheme

$$\begin{aligned} \Lambda\, z(x,t) &\equiv \left\{\varepsilon\,\delta_{\overline{x}\widehat{x}} - \delta_{\overline{t}}\right\} z(x,t) = f(x,t), \quad (x,t)\in G_h, \\ z(x,t) &= \varphi(x,t), \quad (x,t)\in S_h. \end{aligned} \qquad (4.4)$$

Here $\delta_{\overline{x}\widehat{x}}\, z(x,t)$ and $\delta_{\overline{t}}\, z(x,t)$ are the first-order and the second-order difference derivatives,

$$\delta_{\overline{x}\widehat{x}}\, z(x,t) = \frac{2}{h^i + h^{i-1}} \left(\delta_x - \delta_{\overline{x}} \right) z(x,t),$$

$x = x^i$, and h^{i-1} and h^i are the left and right "arms" of the three-point stencil on G_h (of the operator $\delta_{\overline{x}\widehat{x}}$) centered at the nodes $(x^i, t^j) \in G_h$. The difference scheme (4.4), (4.3) satisfies the maximum principle [15].

Note that the operator Λ on the solution of the boundary value problem (2.2), (2.1) is not ε-uniformly bounded (unlike the problem (2.2) on the set $\overline{G}$ with a fixed lateral boundary).

Along with the solutions of the scheme (4.4), (4.3), we will consider their interpolants that can be constructed in the following way. On the basis of the grid $\overline{G}_h$, we construct a triangulation of the domain $\overline{G}$ and use $z(x,t)$, $(x,t) \in \overline{G}_h$, to construct the interpolant $\overline{z}(x,t)$, $(x,t) \in \overline{G}$. We cover the domain $\overline{G}$ by elementary rectangles, irregular quadrangles (with the sides not parallel to the coordinate axes), and triangles. Some vertices of the irregular quadrangles and triangles are the nodes belonging to the set S_h, and one of their sides belongs to the set S^L. We divide the irregular quadrangles into rectangles and irregular triangles. On the lines $t = t^j$, $t^j \in \overline{\omega}_0$, using the values $z(x,t)$, $(x,t) \in \overline{G}_h$, we construct linear (with respect to x) interpolants $\widetilde{z}(x,t)$, $(x,t) \in \overline{G}$, $t \in \overline{\omega}_0$. All the rectangles are partitioned into triangular elements by their diagonals. These regular and irregular triangular elements form a triangulation of the domain $\overline{G}$. On the triangular elements, we construct *linear interpolants* on the basis of the values $\widetilde{z}(x,t)$ at the vertices of the triangular elements on the sets $t = t^j$, $t^j \in \overline{\omega}_0$. The interpolant

$$\overline{z}(x,t) = \overline{z}_{(4.5)}(x,t;\, z(\cdot), \overline{G}_h), \quad (x,t) \in \overline{G} \tag{4.5}$$

is then constructed (see, e.g., [9]).

For the solution of the difference scheme (4.4), (4.3), we have the bound

$$|u(x,t) - z(x,t)| \le M \left(\varepsilon + N^{-1} + N_0^{-1} \right)^{-1} \left[N^{-1} + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h, \tag{4.6}$$

where $\overline{G}_h = \overline{G}_{h(4.3)}$. On the grid (generated by the grid $\overline{G}_{h(4.2)}$)

$$\overline{G}_h = \overline{G}_h^{\,u} = \overline{G}_h\big(\overline{G}^{\,bu}_{h(4.2)}\big), \tag{4.7}$$

i.e., the grid $\overline{G}_{h(4.3)}\big(\overline{G}^{\,b}_{h(4.1)}\big)$, where $\overline{G}^{\,b}_{h(4.1)}$ is the grid $\overline{G}^{\,bu}_{h(4.2)}$, we obtain the bound

$$|u(x,t) - z(x,t)| \le M \left[\left(\varepsilon + N^{-1} \right)^{-2} N^{-2} + \left(\varepsilon + N_0^{-1} \right)^{-1} N_0^{-1} \right], \quad (x,t) \in \overline{G}_h, \tag{4.8}$$

which is unimprovable with respect to N, N_0, and ε.

The schemes (4.4), (4.3) and (4.4), (4.7) converge under the unimprovable condition $N^{-1},\ N_0^{-1} \ll \varepsilon$, or more precisely,

$$\varepsilon^{-1} = o\big(\min[N,\, N_0] \big), \quad N,\, N_0 \to \infty. \tag{4.9}$$

The convergence defect of the scheme (4.4) on the grids (4.3) and (4.7) is equal to unity. In the case of the difference scheme (4.4), (4.3), the function $\overline{z}(x,t)$, $(x,t) \in \overline{G}$, satisfies the bound

$$|u(x,t) - \overline{z}(x,t)| \le M\left(\varepsilon + N^{-1} + N_0^{-1}\right)^{-1}\left[N^{-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}. \quad (4.10)$$

In the case of the scheme (4.4), (4.7), we have the error bound

$$|u(x,t) - \overline{z}(x,t)| \le M\left[\left(\varepsilon + N^{-1}\right)^{-2} N^{-2} + \left(\varepsilon + N_0^{-1}\right)^{-1} N_0^{-1}\right], \ (x,t) \in \overline{G}, \quad (4.11)$$

which is unimprovable with respect to N, N_0, and ε.

Thus, the rate of convergence of the solutions to the difference scheme (4.4), (4.3) (scheme (4.4), (4.7)) and their interpolants are of the same order.

The *optimal order* (with respect to P_0) of convergence rate of the scheme on the grid (4.7) is obtained under the condition $N^2 \approx \varepsilon^{-1} N_0$. In this case, we have the unimprovable (with respect to P_0 and ε) bound

$$|\, u(x,t) - \overline{z}(x,t)\,| \le M\left(\varepsilon^{4/3} + P_0^{-2/3}\right)^{-1} P_0^{-2/3}, \quad (x,t) \in \overline{G}, \quad (4.12)$$

where P_0 is the number of mesh points in $\overline{G}_h$ in the case of the scheme (4.4), (4.7), and $P_0 \approx N\,N_0$.

In the case of the scheme (4.4), (4.3), the least right-hand side in the bound (4.10) is obtained under the condition $N \approx N_0$. Thus, the solution of the scheme (4.4), (4.3) satisfies the bound

$$|\, u(x,t) - \overline{z}(x,t)\,| \le M\left(\varepsilon + P_0^{-1/2}\right)^{-1} P_0^{-1/2}, \quad (x,t) \in \overline{G}, \quad (4.13)$$

which is weaker in comparison with the bound (4.12).

The condition $P_0^{-1/2} \ll \varepsilon$, or more precisely,

$$\varepsilon^{-1} = o\big(P_0^{1/2}\big), \quad P_0 \to \infty \quad (4.14)$$

is necessary and sufficient for the convergence of the scheme (4.4) on the grids (4.3) and (4.7) when the order of convergence rate is optimal with respect to P_0. The convergence defect of these schemes in P_0 and ε is equal to unity.

Theorem 2. *Let the solution of the boundary value problem* (2.2)*,* (2.1) *satisfy the a priori estimates* (3.4) *and* (3.5) *with* $K = 4$*. Then the condition* (4.9) (*the condition* (4.14)) *is necessary and sufficient for the convergence of the scheme* (4.4) *on the grids* (4.3)*,* (4.7) (*on the grids* (4.3)*,* (4.7) *with the optimal order of convergence rate with respect to* P_0)*. The grid solutions satisfy the bounds* (4.6)*,* (4.8)*,* (4.10)*–*(4.13)*.*

5 Construction of ε-Uniform and Almost ε-Uniform Approximations to Solutions of the Problem (2.2), (2.1)

In this section we consider some specific features related to a triangulation of the domain $\overline{G}$ that arise in the construction of ε-uniform approximations to solutions of the singularly perturbed problem (2.2), (2.1). In our considerations of the approximations, we will use an analog of Kolmogorov's widths [1, 2].

5.1. Let $\mathcal{U}$ be a set of solutions in the class of boundary value problems (2.2), (2.1) (defined by the conditions (2.3)). We are interested in the approximation of $\mathcal{U}$ in the space X, that is, the set of continuous functions with the maximum norm. The solutions are assumed to be sufficiently smooth on $\overline{G}$ for fixed values of the parameter ε; the solutions and their components in the representation (3.1) satisfy the estimates (3.4) and (3.5).

Let us describe approximations to the solutions. Let $\overline{G}^{\,h}$ be a finite set of points (we say, *the grid*) on $\overline{G}$. The grids $\overline{G}^{\,h}$ may be both structured (generated by some regular family of lines) and unstructured. The number of *nodes* in the grid $\overline{G}^{\,h}$ on $\overline{G}$ is denoted by P; $\overline{G}^{\,h} = \overline{G}^{\,h}(P)$. Let T_P be a *triangulation* (partition) of the set $\overline{G}$ *generated by the grid* $\overline{G}^{\,h}$ (see, e.g., [9]); we assume that the mesh points in $\overline{G}^{\,h}$ are the vertices of the triangular elements, where the triangle sides are line segments that pass through the nodes of $\overline{G}^{\,h}$ if at least one of the endpoints of a triangle side belongs to G^h, or are segments of curves if both of the endpoints belong to the boundary $\overline{S}^L$; here $G^h = \overline{G}^{\,h} \bigcap G$. Let some grid function $u^h(x,t)$, $(x,t) \in \overline{G}^{\,h}$, be defined on the set $\overline{G}^{\,h}$. By $\overline{u}^{\,h}(x,t)$, $(x,t) \in \overline{G}$, we denote the piecewise-linear interpolant that is linear on each triangle and is constructed using the values of $u^h(x,t)$ at the vertices of the triangular elements. The set of such piecewise-linear interpolants for the fixed triangulation T_P is denoted by U_P^h. The set of all feasible grid sets $\overline{G}^{\,h}$ (however, with the number of nodes equal to P on $\overline{G}$) and of triangulations T_P based on them will be denoted by $\mathcal{T}_P$; we say that $\mathcal{T}_P$ is the *set of triangulations* of the domain $\overline{G}$. The classes of feasible grid sets and triangulations on them are specified in detail below. This set of triangulations $\mathcal{T}_P$ and the set of interpolants U_P^h (for each triangulation in $\mathcal{T}_P$) approximate the *space* X. Define the *width* $d_P(\mathcal{U},\, X)$ by

$$d_P(\mathcal{U},\, X) = \inf_{\mathcal{T}_P} \sup_{u \in \mathcal{U}} \inf_{\overline{u}^{\,h} \in U_P^h} \| \, u - \overline{u}^{\,h} \, \|, \tag{5.1}$$

where $\| \cdot \|$ is the maximum norm in $C(\overline{G})$. A definition of Kolmogorov's width can be found, for example, in [1, Chapter 3]. The quantity $d_P(\mathcal{U},\, X)$ is the *error of the optimal approximation of the set* $\mathcal{U}$ *in the space* X using a grid with P nodes, or, briefly, the *error of the optimal approximation*.

Definitions. Let $d_P^i(\mathcal{U},\, X) = d_P(\mathcal{U},\, X;\, \overline{G}_i^h)$, $i = 1, 2$ be the widths induced by two families of grids $\overline{G}_i^h = \overline{G}_i^h(P)$, $i = 1, 2$. When the widths $d_P^i(\mathcal{U},\, X)$ satisfy the bound

$$m\, d_P^1(\mathcal{U},\, X) \le d_P^2(\mathcal{U},\, X) \le M\, d_P^1(\mathcal{U},\, X), \quad P \ge M_1,$$

uniformly with respect to P for some sufficiently large M_1, we say that the widths $d_P^1(\mathcal{U},\, X)$ and $d_P^2(\mathcal{U},\, X)$ are *equivalent*.

By $\rho_1(T_P^j)$ and $\rho_2(T_P^j)$, we denote the radii of the inscribed and circumscribed circles for the triangular element T_P^j in the triangulation T_P, $j =$

$1, \ldots, J$, where $J = J(P)$ is the number of triangular elements in T_P (we take $J \approx P$). The triangulation T_P is said to be *isotropic* if the condition

$$\rho_1^{-1}(T_P^j)\, \rho_2(T_P^j) \le M, \quad j = 1, \ldots, J,$$

holds, however, the quantities $\rho_1^{-1}(T_P^j)$, $\rho_2(T_P^j)$ and *the anisotropy coefficient* $\eta(T_P^j) = \rho_1^{-1}(T_P^j)\, \rho_2(T_P^j)$ for each element T_P^j can differ significantly from element to element. The triangulation T_P is called *anisotropic* (with *the anisotropy coefficient* $\eta \ge M_0$, where the lower threshold M_0 may be sufficiently large) if

$$\eta \equiv \sup_{j=1,\ldots,J} \eta(T_P^j) \ge M_0,$$

and the constant M_0 does not depend on the parameters ε, P. We assume that the set $\mathcal{T}_P$, as well as the width $d_P(\mathcal{U},\, X)$, is determined by η; thus,

$$T_P = T_P(\eta), \quad \mathcal{T}_P = \mathcal{T}_P(\eta), \quad d_P(\mathcal{U},\, X) = d_P(\mathcal{U},\, X;\, \eta).$$

Isotropic and anisotropic triangulations on subsets of $\overline{G}$ can be defined in a similar way. When the widths are considered on the subset $\overline{G}^{\,0} \subset \overline{G}$ (in this case, we denote the width by $d_P(\mathcal{U},\, X;\, \overline{G}^{\,0})$), the quantity $\|\, u - \overline{u}^{\,h} \,\|$ in (5.1) is computed using only the triangular elements that belong entirely to $\overline{G}^{\,0}$.

A triangulation element satisfying the condition

$$\eta(T_P^j) \to \infty \;\text{ as }\; P \to \infty \;\text{ and/or }\; \varepsilon \to 0,$$

is called *essentially anisotropic*; a triangulation T_P containing such elements is said to be *essentially anisotropic*.

The width $d_P(\mathcal{U},\, X)$ tends to zero as $P \to \infty$, however, this convergence to zero is not ε-uniform. In general, the approximations converge as $P \to \infty$ only for certain relationships between P and ε.

Let the width $d_P(\mathcal{U},\, X)$ satisfy the upper bound

$$d_P(\mathcal{U},\, X) \le M\, \lambda \left(\varepsilon^{-\nu}\, P^{-1/2}\right), \quad P \to \infty, \quad \varepsilon \in (0,\, 1], \tag{5.2}$$

that is similar to the estimate (2.6) for the difference scheme. As in the convergence of solutions to difference schemes, one can define *the convergence of the width* (the error of the optimal approximation) *with defect* ν (of the ε-uniform convergence) with respect to the values of P and ε and also its *ε-uniform and almost ε-uniform convergence*. In the case of almost ε-uniform convergence, the value ν controls the triangulations T_P. The most interesting approximations of the set $\mathcal{U}$ are those that converge (if possible) with the minimal defect and, in particular, those that converge ε-uniformly.

5.2. Consider a bound on the width when the moving boundary S_1^L is a line segment

$$\beta_1(t) = v_1\, t, \quad t \in [0,\, T], \quad v_1 \in [v_0,\, v^0], \tag{5.3}$$

where $v_0 = v_{0(2.3)}$, $v^0 = v^0_{(2.3)}$. In the case of isotropic triangulations of the domain $\overline{G}$, the width satisfies the lower bound

$$d_P(\mathcal{U},\, X) \;\ge\; m(1 + \varepsilon P)^{-1}. \tag{5.4}$$

On the set

$$\overline{G}^{\,0} = \overline{G}_1(\rho_1), \quad \rho_1 = M\,\varepsilon, \tag{5.5}$$

where $G_1(\rho_1) = \big\{(x,t) \,:\, x \in \big(\beta_1(t),\, \beta_1(t) + \rho_1\big),\ t \in (0,T]\,\big\}$ is the right ρ_1-neighbourhood of the set S_1^L, we obtain the bound

$$d_P\big(\mathcal{U},\, X;\, \overline{G}^{\,0}_{(5.5)}\big) \,\geq\, m(1+\varepsilon P)^{-1}, \tag{5.6}$$

which is unimprovable with respect to P and ε.

In the case of anisotropic triangulations, taking into account the explicit form of the main term of the asymptotic expansion (in powers of ε) of the singular component in the representation (3.1), under the condition (5.3) we find the bound

$$d_P(\mathcal{U},\, X) \,\geq\, m\,\min\Big\{\big(\varepsilon\,\eta + (\varepsilon\,\eta)^{-1}\big)\,P^{-1},\, 1\Big\}. \tag{5.7}$$

On the set $\overline{G}^{\,0}_{(5.5)}$, we have the bound

$$d_P(\mathcal{U},\, X;\, \overline{G}^{\,0}) \,\geq\, m\,\min\Big\{\big(\varepsilon\,\eta + (\varepsilon\,\eta)^{-1}\big)\,P^{-1},\, 1\Big\}, \tag{5.8}$$

which is unimprovable with respect to P, ε, and η.

5.3. Assume that the condition (5.3) is fulfilled. The bounds (5.6) and (5.8), under the condition

$$M_0 \leq \eta \leq M_1, \quad \text{where the constant } M_1 \text{ is independent of } P \text{ and } \varepsilon, \tag{5.9}$$

show that the convergence defect of the error of the optimal approximation on the set $\overline{G}^{\,0}_{(5.5)}$ is equal to 2^{-1}. Therefore, the convergence defect of the widths $d_P(\mathcal{U},\, X)$ on an isotropic triangulation and on an anisotropic triangulation under the condition (5.9) is not less than 2^{-1}.

By virtue of the bound (5.8), the condition (of the essential anisotropy of the triangulations)

$$\begin{aligned} &\eta = \eta(\varepsilon,\, P), \quad \eta(\varepsilon,\, P) \to \infty \ \text{ for } \ P \to \infty \ \text{ and/or } \ \varepsilon \to 0; \\ &P \to \infty, \quad \varepsilon \in (0,1], \end{aligned} \tag{5.10}$$

is necessary for the defect (of convergence of the width on the anisotropic triangulation T_P) to be less than 2^{-1}, and also for ε-uniform or almost ε-uniform convergence.

According to the condition (5.10) (necessary for convergence of the width with defect less than 2^{-1}) and from the bounds (5.7) and (5.8) (unimprovable with respect to P, ε, and η), we define η by

$$\eta = \eta(\varepsilon,\, P) \equiv \varepsilon^{-1}\,\eta_0(P), \quad \text{where} \ \ \eta_0(P) \to \infty \ \text{ for } \ P \to \infty. \tag{5.11}$$

Then the additional condition $\eta_0(P)\,P,\ \eta_0^{-1}(P)\,P \gg 1$; more precisely,

$$\eta_0(P),\, \eta_0^{-1}(P) = o(P), \quad P \to \infty, \tag{5.12}$$

is necessary for the ε-uniform convergence of the width $d_P(\mathcal{U},\, X;\, \overline{G}^{\,0})$ on the set $\overline{G}^{\,0} = \overline{G}^{\,0}_{(5.5)}$. Under the condition (5.11), taking into account the a priori bounds of Theorem 1, we obtain the ε-uniform upper bound

$$d_P(\mathcal{U},\, X;\, \overline{G}^{\,0}) \,\leq\, M\, P^{-1} \big[\eta_0(P) + \eta_0^{-1}(P)\big], \tag{5.13}$$

this bound is unimprovable with respect to P.

For the width $d_P(\mathcal{U},\, X)$, under the condition (5.11) we obtain the ε-uniform bound

$$d_P(\mathcal{U},\, X) \,\leq\, M\, \min \Big[\, P^{-1}\, \ln P \,\big[\,\eta_0(P) +\, \eta_0^{-1}(P)\big];\, 1\Big]. \tag{5.14}$$

Under the additional condition $P^{-1}\, \ln P \,\big[\,\eta_0(P){+}\,\eta_0^{-1}(P)\big] \ll 1$; more precisely,

$$\eta_0(P),\, \eta_0^{-1}(P) = o(P\, \ln^{-1} P), \quad P \to \infty, \tag{5.15}$$

which is slightly stronger than the condition (5.12), the width $d_P(\mathcal{U},\, X)$ converges ε-uniformly.

Theorem 3. *Let the components of the solution of the boundary value problem* (2.2)*,* (2.1) *in the representation* (3.1) *satisfy the a priori estimates* (3.4) *and* (3.5)*, where $K = 2$. In the case of the condition* (5.3)*, the condition* $\{(5.11),$ $(5.12)\}$ *is necessary and the condition* $\{(5.11),\, (5.15)\}$ *is sufficient for the ε-uniform convergence* (*as $P \to \infty$*) *of the width $d_P(\mathcal{U},\, X)$ for the anisotropic triangulation* (5.11)*. Under the condition* (5.11)*, the widths $d_P(\mathcal{U},\, X;\, \overline{G}^{\,0})$ and $d_P(\mathcal{U},\, X)$ satisfy the bounds* (5.13) (*which is unimprovable with respect to P*) *and* (5.14)*, respectively.*

Remark 1. In the case of the condition (5.3), the necessary condition $\{(5.11),$ $(5.12)\}$ and the sufficient condition $\{(5.11),\, (5.15)\}$ for the ε-uniform convergence of the width, are almost the same. From the unimprovable bound (5.13) and the representation (5.11) for the value η, it follows that the triangulation elements on the set $\overline{G}^{\,0}$ are essentially anisotropic if the width on $\overline{G}^{\,0}$ converges with defect $\nu < 2^{-1}$.

5.4. Assume that the following condition holds:

$$\textit{the set } S_1^L \textit{ is a segment of a smooth curve} \tag{5.16}$$

with a bounded nonzero curvature, and let the triangulation T_P be anisotropic and satisfy the condition

all the sides of the triangular elements in the triangulation are line segments. (5.17)

On the triangulations $T_P = T_P(\eta)$ having anisotropy η, we define the width $d^*_P(\mathcal{U},\, X)$ by

$$d^*_P(\mathcal{U},\, X) = \inf_{\eta}\; d_P(\mathcal{U},\, X;\, \eta). \tag{5.18}$$

We say that the width $d^*_P(\mathcal{U},\, X)$ is *optimal with respect to the anisotropy* on the triangulations $T_P(\eta)$, or, briefly, we say the *optimal width*.

In the case of the conditions (5.16) and (5.17), taking into account the explicit form of the main term of the component $W_{(3.1)}(x,\, t)$, we obtain the following unimprovable bound for the width $d_P^*(\mathcal{U},\, X)$ considered on $\overline{G}^{\,0}_{(5.5)}$:

$$d_P^*(\mathcal{U},\, X;\, \overline{G}^{\,0}) \,\geq\, m\,(1+\varepsilon^{1/2}P)^{-1}. \tag{5.19}$$

Theorem 4. *Let the assumption of Theorem 3 holds. Then, under the conditions* (5.16) *and* (5.17)*, the width* $d^*_{P(5.18)}(\mathcal{U},\, X;\, \overline{G}^{\,0})$ *satisfies the bound* (5.19).

Remark 2. By virtue of the bound (5.19), the convergence defect of the width $d_P^*(\mathcal{U},\, X;\, \overline{G}^{\,0})$ in the case of a curvilinear boundary S_1^L is 4^{-1}. Thus, for the width $d_P^*(\mathcal{U},\, X)$ in the case of the triangulations generated by the triangular elements satisfying condition (5.17) and the interpolants $\overline{u}^{\,h}(x,t)$ that are linear on the elements T_P^j, a convergence defect less than 4^{-1} cannot be achieved in the case of condition (5.16).

Remark 3. It follows from the considerations given above that, in order to construct optimal approximations whose widths have a defect of convergence less than 4^{-1}, it is necessary to use triangulations T_P with curvilinear triangular elements T_P^j and/or nonlinear interpolants constructed on T_P^j from the values of the function $u^h(x,t)$.

Remark 4. Theorem 4 implies that, in the case of the condition (5.16), a convergence defect less than 4^{-1} cannot be achieved also for the difference schemes constructed by the classical approximation of the boundary value problem (2.2), (2.1) on (reasonable) meshes generating a triangulation that satisfies the condition (5.17).

6 Difference Scheme on a Grid Adapted in the Moving Boundary Layer

We are interested in the difference schemes for which the interpolants of the grid solutions constructed on the triangulation elements generated by the grid nodes are convergent ε-uniformly. The results given in Section 5 imply that the optimal approximations of the boundary value problem (2.2), (2.1) constructed on the basis of regular anisotropic triangulations and linear interpolants converge with a defect not lower that 4^{-1} in the case of a curvilinear boundary S_1^L. Hence, it follows that, for the solutions of the scheme (4.4) on the grids generated by rectangular (base) grids, a convergence defect of the interpolants of these grid solutions less than 4^{-1} is unachievable. Therefore, in the case of a curvilinear boundary S_1^L, for the convergence of the interpolants of grid solutions with defect lower than 4^{-1} (or ε-uniformly), it is necessary to use grids fitted to the boundary S_1^L that generate essentially anisotropic triangulation elements. These conditions, which are necessary for the ε-uniform approximations of the widths, are used in the construction of an ε-uniformly convergent scheme.

6.1. To construct a special scheme that converges ε-uniformly, we use the following approach. After passing to the variables ξ, t, $\xi = \xi(x,\, t)$, the problem (2.2), (2.1) is transformed into the problem (3.3), (3.3) in a domain with fixed lateral boundaries. For this problem, the optimal approximations of its solution converge ε-uniformly in the case of the triangulations based on the grids that are piecewise uniform with respect to ξ and uniform with respect to t and on the interpolants that are linear on the triangulation elements. Having constructed an ε-uniformly convergent scheme for the problem (3.3), (3.3) (in such a "standard" difference scheme, the linear interpolant based on its grid solutions converges ε-uniformly) and then having returned to the initial variables, we obtain an ε-uniformly convergent scheme for the problem (2.2), (2.1); for this scheme, the interpolant is written in terms of the grid solutions but is no longer linear in x and t on the triangulation elements.

In terms of the initial variables, the resulting grids are no longer rectangular (the distribution of the grid nodes is adapted to the moving boundary S_1^L). Generally, this implies some inconvenience in the construction of grid domains and in the numerical solution of the problem. However, such a scheme may be constructed only in a small neighbourhood of the boundary S_1^L; outside this neighbourhood, rectangular (in the initial variables) grids and classical grid approximations of the problem may be used.

According to this approach, we pass from the problem (2.2), (2.1) to the problem (3.3), (3.3) for which we construct a scheme on *a priori* condensing grids. On the set $\overline{\widetilde{G}}$, we introduce the rectangular grid

$$\overline{\widetilde{G}}_h = \overline{\widetilde{\omega}}_1 \times \overline{\omega}_0, \tag{6.1}$$

where $\overline{\widetilde{\omega}}_1$ and $\overline{\omega}_0$ are meshes on the intervals $\overline{\widetilde{D}} = [0, d]$ and $[0, T]$, respectively; $\overline{\omega}_0 = \overline{\omega}_{0(4.2)}$, $\overline{\widetilde{\omega}}_1$ is a mesh with an arbitrary distribution of nodes satisfying the condition $h_\xi \leq M N^{-1}$, where $h_\xi = \max_i h_\xi^i$, $h_\xi^i = \xi^{i+1} - \xi^i$, ξ^i, $\xi^{i+1} \in \overline{\widetilde{\omega}}_1$; and $N+1$ is the number of mesh points in $\overline{\widetilde{\omega}}_1$.

To solve the problem (3.3), (3.3), we use the difference scheme

$$\begin{cases} \widetilde{\Lambda}\, Z(\xi,t) \equiv \Big\{ \varepsilon A(\xi,t)\, \delta_{\overline{\xi}\widehat{\xi}} + B(\xi,t)\, \delta_\xi - \delta_{\overline{t}} \Big\} Z(\xi,t) = \widetilde{f}(\xi,t), \ (\xi,t) \in \widetilde{G}_h, \\ Z(\xi,t) = \widetilde{\varphi}(\xi,t), \quad (\xi,t) \in \widetilde{S}_h, \end{cases} \tag{6.2}$$

where $\widetilde{G}_h = \widetilde{G} \bigcap \overline{\widetilde{G}}_h$, $\widetilde{S}_h = \widetilde{S} \bigcap \overline{\widetilde{G}}_h$, and $\delta_{\overline{\xi}\widehat{\xi}} Z(\xi,t)$ and $\delta_\xi\, Z(\xi,t)$, $\delta_{\overline{t}}\, Z(\xi,t)$ are the second and the first difference derivatives.

Furthermore, we consider the well-known "standard" piecewise-uniform grid (see, for example, [4, 5, 18]).

6.2. On the set $\overline{\widetilde{G}}$, we define the "standard" grid $\overline{\widetilde{G}}_h^S$ condensing in a neighbourhood of the boundary layer:

$$\overline{\widetilde{G}}_h^S = \overline{\widetilde{\omega}}_1^S \times \overline{\omega}_0, \tag{6.3}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(6.1)}$ and $\overline{\widetilde{\omega}}_1^S = \overline{\widetilde{\omega}}_1^S(\sigma)$ is a piecewise-uniform mesh. The meshsizes of $\overline{\widetilde{\omega}}_1^S$ are constant on the intervals $[0, \sigma]$ and $[\sigma, d]$ and equal to $h_{(1)} = 2\,\sigma\, N^{-1}$ and $h_{(2)} = 2\,[d - \sigma]\, N^{-1}$, respectively; the value σ is chosen so as to satisfy the condition $\sigma = \sigma(\varepsilon, N) = \min\left[2^{-1}d,\ m^{-1}\,\varepsilon \ln N\right]$, where m is an arbitrary number in the interval $(0, m^0)$, and $m^0 = d^{-1}\, m_{1(2.3)}\, v_{0(2.3)}$.

The scheme (6.2), (6.3) converges ε-uniformly with the error bound

$$|\widetilde{u}(\xi,t) - Z(\xi,t)| \le M\ \left[N^{-1}\ln N + N_0^{-1}\right], \quad (\xi,t) \in \overline{\widetilde{G}}_h.$$

For the interpolant $\overline{Z}(\xi,t)$, $(\xi,t) \in \overline{\widetilde{G}}$, which is linear in ξ and t on the triangular elements of the triangulation generated by the grid $\overline{\widetilde{G}}_h$, we have the bound

$$\left|\widetilde{u}(\xi,t) - \overline{Z}(\xi,t)\right| \le M\ \left[N^{-1}\ln N + N_0^{-1}\right], \quad (\xi,t) \in \overline{\widetilde{G}}.$$

In the variables x and t, the grid

$$\overline{\widetilde{G}}_{h\,\xi^{-1}} = \left\{\overline{\widetilde{G}}_{h(6.3)}\right\}_{\xi^{-1}} \tag{6.4}$$

is not a tensor product of meshes in x and t. This grid is uniform with respect to t and is piecewise uniform with respect to x for $t = t^j$, $t^j \in \overline{\omega}_0$. Passing to the variables x and t in the scheme (6.2), we come to the scheme

$$\begin{aligned}\widetilde{\Lambda}_{\xi^{-1}}\, Z_{\xi^{-1}}(x,t) \equiv \varepsilon\, A_{\xi^{-1}}(x,t)\left\{\delta_{\overline{\xi}\widehat{\xi}}\, Z(\xi,t)\right\}_{\xi^{-1}} + B_{\xi^{-1}}(x,t)\left\{\delta_{\xi}\, Z(\xi,t)\right\}_{\xi^{-1}} \\ - \left\{\ \delta_{\overline{t}}\, Z(\xi,t)\right\}_{\xi^{-1}} = f(x,t), \quad (x,t) \in \widetilde{G}_{h\,\xi^{-1}}, \qquad (6.5)\end{aligned}$$

$$Z_{\xi^{-1}}(x,t) = \varphi(x,t), \quad (x,t) \in \widetilde{S}_{h\,\xi^{-1}}.$$

The function $Z^*(x,t) = Z_{\xi^{-1}}(x,t)$, $(x,t) \in \overline{G}_h^*$, where $\overline{G}_h^* = \overline{\widetilde{G}}_{h\,\xi^{-1}}$, i.e., the solution of the difference scheme (6.5), (6.4), satisfies the bound

$$|u(x,t) - Z^*(x,t)| \le M\ \left[N^{-1}\ln N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h^*. \tag{6.6a}$$

For the interpolant $\overline{Z}^*(x,t) = \left\{\ \overline{Z}(\xi,t)\right\}_{\xi^{-1}}$, $(x,t) \in \overline{G}$, which is obtained from the function $\overline{Z}(\xi,t)$ by passing to the variable x and t, we have a similar bound:

$$\left|u(x,t) - \overline{Z}^*(x,t)\right| \le M\ \left[N^{-1}\ln N + N_0^{-1}\right], \quad (x,t) \in \overline{G}. \tag{6.6b}$$

Theorem 5. *Let the assumption of Theorem 2 be fulfilled. Then the difference scheme* (6.5), (6.4) *approximating the boundary value problem* (2.2), (2.1) *converges ε-uniformly. The grid solution $Z^*(x,t)$, $(x,t) \in \overline{G}_h^*$, and the interpolant $\overline{Z}^*(x,t)$, $(x,t) \in \overline{G}$, satisfy the bounds* (6.6).

Remark 5. For the solution of the difference scheme (6.5), (6.4), we have the bound (see (6.6))

$$|u(x,t) - Z^*(x,t)| \le M\ \left[N^{-1}\ln N + N\,P^{-1}\right], \quad (x,t) \in \overline{G}_h^*,$$

where $P = N\,N_0$. Under the condition $N \approx N_0$, which is natural for regular problems, we obtain the ε-uniform bound

$$|u(x,t) - Z^*(x,t)| \le M\,P^{-1/2}\ln P, \quad (x,t) \in \overline{G}_h^*. \tag{6.7}$$

7 Remarks and Generalizations

In this section we give some remarks and generalizations related to application of the technique based on the widths for investigation of ε-uniformly convergent difference schemes.

7.1. From discussion of Section 3–5, one obtains the following variant of our approach to the construction of ε-uniformly convergent difference schemes. The problem (2.2), (2.1), by a change of variables, is transformed to a problem with stationary boundaries. For the new problem, a scheme is constructed that converges ε-uniformly. Next, we find the solution of this scheme and return to the original variables. Such an approach was used in [12] for a parabolic convection-diffusion equation in a domain with moving boundaries.

7.2. To construct ε-uniformly convergent schemes for problem (2.2), (2.1), one can use *the domain decomposition method* on overlapping subdomains. In the subdomain that includes the boundary layer, a finite difference scheme is constructed applying the grid constructs given in Section 5. In the subdomain not including boundary layer, a classical finite difference scheme based on uniform meshes is constructed similar to those in Section 4. One can show that in the case when the minimal width of the subdomain overlaps is not less than value of the parameter ε and the maximum of the step-sizes in x and t, the domain decomposition scheme converges ε-uniformly.

7.3. The approach based on widths is applied to construct ε–uniformly convergent finite difference scheme for *elliptic and parabolic equations* in domains with *curvilinear boundaries*. Analysis of widths allows us to determine conditions that are necessary and (for additional assumptions) sufficient for ε-uniform convergence of the finite difference schemes.

In [23], *widths* were studied for *a class of solutions to boundary value problems* of elliptic reaction–diffusion equations in two dimensional domains with curvilinear boundaries. In these problems, for ε–uniform convergence (in the maximum norm) of widths defined on grid sets, it is *necessary* that these grid sets be condensed in a neighbourhood of the boundary layer and adapted to the domain boundary. It is not difficult to realize such adaptation of the grid sets if we pass to a local coordinate system in which the piece of the boundary becomes a line part. In the new coordinate system, on a simplest grid, which are uniform along the boundary and piecewise-uniform along the normal to the boundary, one can derive *sufficient conditions close to necessary* for ε-uniform convergence of widths. The transition point in the piecewise-uniform meshes depends both on the parameter ε and the value N_1, i.e., the number of nodes in the mesh along the normal to the boundary. It is necessary for these piecewise-uniform meshes to be consistent with the boundary in its σ_0^*-neighbourhood, where

$$\sigma_0^* = \sigma_{(1)} + \sigma_{(2)}, \quad \sigma_{(1)} \approx \varepsilon\lambda(N_1), \quad \sigma_{(2)} \approx N_2^{-2};$$

$$\lambda(N_1) \to \infty, \quad N_1^{-1}\lambda(N_1) \to 0, \quad \textit{for} \;\; N_1, N_2 \to \infty, \quad N_1 N_2 \approx P.$$

Here P is the number of nodes in the grid set. Outside the σ_0^*-neighbourhood, the widths converge ε-uniformly already on grid sets with isotropic triangulation, e.g., generated by uniform grids. In [23], for a differential problem local difference schemes were constructed on such type grids, adapted in a neighbourhood of the boundary. These schemes possess the required properties such as approximation and stability. Furthermore, using the domain decomposition method based on the local approximations of the problem, ε-uniformly convergent finite difference schemes were constructed for elliptic reaction-diffusion equations in a domain with a curvilinear boundary.

Similar approach is realized in [18] to construct ε-uniformly convergent finite difference schemes for elliptic and parabolic problems for reaction-diffusion and convection-diffusion equations in *n-dimensional domains with* smooth and *piecewise smooth boundaries*.

7.4. *The use of widths* allows us to justify *applicability of the fitted operator method* in the construction of ε–uniformly convergent difference scheme for parabolic convection–diffusion and reaction–diffusion problems. For an initial-boundary value problem in a domain with stationary boundaries, the singular component of the solution to the convection-diffusion problem is the regular boundary layer (described by an ordinary differential equation) and to the reaction-diffusion problem it is the parabolic boundary layer (described by a parabolic equation). The main term in an expansion of the singular component, i.e., *the regular layer*, is defined by only *one coefficient-parameter*, namely, the ratio of coefficients to the second- and first-order derivatives in x. The main term in an expansion of the singular component, i.e., *the parabolic layer*, is an infinite sum of singular components (we say, elementary singular components) that are defined by both *the coefficient-parameter* which is the ratio of coefficients to the second-order derivative in x and the first-order derivative in t, and by *the coefficients of the Taylor expansion* to the boundary function.

Under the condition that the problem solution does not involve the main term of the singular component (i.e., the first term of the expansion to the boundary layer function), the solutions of the initial-boundary value problems for convection-diffusion and reaction-diffusion equations generate the set $\mathcal{U}^0$, i.e., the set of regular solutions, whose widths $d_P(\mathcal{U}^0, X)$ converge ε–uniformly for finite values of anisotropy η to the domain triangulation.

For *the fitted operator method* applied to *the convection-diffusion problem*, the main term of the singular component is the solution of a homogeneous difference equation. At the same time, the regular component of the solution to the finite difference scheme converges ε-uniformly to the regular component of the solution to the differential problem that implies the ε-uniform convergence of the fitted operator scheme.

When *the fitted operator method* is applied to *the reaction-diffusion problem*, by choosing a finite number of the fitted coefficients in a scheme, it is possibly to attain that only some elementary singular components of the main term of the singular component (i.e., the parabolic-layer function) could be the solution of the homogeneous difference equation.

The remainder elementary singular components, as well as the regular components of the solutions, generate the set $\mathcal{U}^1$, i.e., the set of the singular solutions, whose widths $d_P(\mathcal{U}^1, X)$ do not already converge ε-uniformly for finite values of the anisotropy η to the domain triangulations. Hence, for the reaction-diffusion problems with the boundary parabolic layers, there exist no schemes of the fitted operator method that converge ε-uniformly.

The proof of similar statement in [11, 17, 18] is carry out *in other way* unlike given here. In [11, 17, 18], the direct check justifies that, for a finite number of the fitted coefficients, there exist elementary singular components in the singular term of the problem solution such that are not approximated ε–uniformly by discrete solutions.

7.5. When constructing ε-uniformly convergent fitted operator schemes for parabolic reaction–diffusion problems with parabolic layers, difficulties are observed for both *the boundary layer* and *the initial layer*. In [19], an initial-boundary value problem was considered for a parabolic reaction-diffusion equation with *perturbation parameters multiplied by the spatial and temporal derivatives*. In this problem, initial, boundary and parabolic layers appear, depending on the relations between the parameters. In [19], using a technique, similar to given in [11, 17, 18], it was shown that, in the presence of the initial parabolic layers, there exist no schemes of the fitted operator method that converge ε-uniformly. But the construction of ε-uniformly convergent schemes of the fitted operator method resolving the initial-boundary parabolic layers, have no difficulties. The same results can be obtained using *a technique based on the widths*.

7.6. In [22], a boundary value problem is considered for an elliptic convection-diffusion equation with a perturbation vector-parameter $\overline{\varepsilon} = \varepsilon$, where $\overline{\varepsilon} = (\varepsilon_1, \varepsilon_2)$. The second-order and the first-order derivatives in the differential equation are multiplied by the component-parameters ε_1^2 and ε_2^2 respectively. Depending on the relations between the component-parameters, regular, parabolic and *hyperbolic* boundary layers appear. In [22], on the basis of a technique for analyzing convergence of the fitted operator schemes for problems with parabolic layers (see, for example, [11, 17, 18]), the conclusion is derived that *in the presence of hyperbolic layers, the fitted operator method is inapplicable* for the construction of ε-uniformly convergent schemes. The same result is obtained naturally when *the width technique* is applied.

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