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UNCERTAINTY PRINCIPLES FOR THE KONTOROVICH-LEBEDEV TRANSFORM

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Abstract. By using classical uncertainty principles for the Fourier transform and composition properties of the Kontorovich-Lebedev transform, analogs of the Hardy, Beurling, Cowling-Price, Gelfand-Shilov and Donoho-Stark theorems are obtained.

Key words: Kontorovich-Lebedev transform, Fourier transform, Laplace transform, modified Bessel functions, Hardy theorem, Cowling-Price theorem, Beurling theorem, Gelfand-Shilov theorem, Donoho-Stark theorem, uncertainty principle.

1 Introduction

The Kontorovich-Lebedev transformation is defined as follows (cf. [9, 15, 18, 19, 20])

$$K_{ix}[f] = \int_0^\infty K_{ix}(y) f(y)\, dy, \; x > 0. \tag{1.1}$$

The kernel of the transformation (1.1) is a particular case of the modified Bessel function $K_\mu(z)$ [4], which in turn, is an independent solution of the differential equation

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2+\mu^2)u = 0.$$

When $\mu = ix, \; x \in \mathbb{R}, z = y > 0$, then $K_{ix}[f]$ is real valued and even function with respect to x. If $f \in L_2(\mathbb{R}_+; y\, dy)$, then $K_{ix}[f] \in L_2(\mathbb{R}_+; x \sinh \pi x\, dx)$ (see [18, 19]), and the Parseval formula holds

$$\int_0^\infty x \sinh(\pi x) \left|K_{ix}[f]\right|^2 dx = \frac{\pi^2}{2} \int_0^\infty |f(y)|^2 y\, dy. \tag{1.2}$$

In this case integral (1.1) converges in the mean square sense and can be written, making necessary truncations at zero and infinity. Moreover, the inverse transform has the form

$$yf(y) = \frac{2}{\pi^2} \int_0^\infty x \sinh \pi x K_{ix}(y) K_{ix}[f]\, dx, \tag{1.3}$$

where integral (1.3) is in the mean square sense with the necessary truncation at infinity.

On the other hand, if $f \in L_1(\mathbb{R}_+; K_0(y)\, dy)$, where $K_0(y)$ is the modified Bessel function of the index zero, then inversion formula (1.3) can be interpreted at each Lebesgue point of f (see in [20]) as

$$yf(y) = \frac{4}{\pi^2} \lim_{\alpha \to \frac{\pi}{2}-} \int_0^\infty x \sinh \alpha x \cosh \frac{\pi x}{2} K_{ix}(y) K_{ix}[f]\, dx. \tag{1.4}$$

If also $K_{ix}[f] \in L_1(\mathbb{R}_+; x \cosh \frac{\pi x}{2}\, dx)$, then we can pass to the limit in (1.4) under the integral sign and we get (1.3) in Lebesgue integrable sense.

The modified Bessel function has the following asymptotic behavior

$$K_\mu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} [1 + O(1/z)], \qquad z \to \infty, \tag{1.5}$$

and near the origin

$$z^{|\mathrm{Re}\mu|} K_\mu(z) = 2^{\mu-1} \Gamma(\mu) + o(1), \quad z \to 0, \ \mu \neq 0, \tag{1.6}$$

$$K_0(z) = -\log z + O(1), \quad z \to 0. \tag{1.7}$$

Meanwhile, when x is restricted to any compact subset of $\mathbf{R}_+$ and τ tends to infinity we have the following asymptotic (see, [18], p. 20)

$$K_{i\tau}(x) = \left(\frac{2\pi}{\tau}\right)^{1/2} e^{-\pi\tau/2} \sin\left(\frac{\pi}{4} + \tau \log \frac{2\tau}{x} - \tau\right) \left[1 + O\Big(\frac{1}{\tau}\Big)\right], \quad \tau \to \infty. \tag{1.8}$$

The modified Bessel function can be represented by the integrals of the Fourier and Mellin types [4, 12, 15, 18, 20], respectively

$$K_\mu(x) = \int_0^\infty e^{-x \cosh u} \cosh \mu u \ du, \tag{1.9}$$

$$K_\mu(x) = \frac{1}{2} \left(\frac{x}{2}\right)^\mu \int_0^\infty e^{-t - \frac{x^2}{4t}} t^{-\mu-1} dt,$$

$$\sinh \frac{\pi\tau}{2} K_{i\tau}(x) = \int_0^\infty \sin(x \sinh u) \sin \tau u du, \tag{1.10}$$

$$\cosh \frac{\pi\tau}{2} K_{i\tau}(x) = \int_0^\infty \cos(x \sinh u) \cos \tau u du.$$

The main aim of the paper is to establish the so-called uncertainty principles for the operator (1.1), which say that a nonzero original and its image under transformation (1.1) cannot be simultaneously too small in the pointwise or integrable decay. This comes as a generalization of the classical Heisenberg uncertainty principle. It was extended to the Fourier transform in [5, 6, 7, 11]. The corresponding principles have been proved also for the Jacobi transform [2, 10], the Y-transform [1], the Dunkl transform [13] and recently for the Hankel transform [14, 17].

The structure of the paper is as follows: in Section 2 we will prove Hardy's type theorem for the Kontorovich-Lebedev transformation, which will give as a corollary the corresponding Hardy uncertainty principle. Section 3 of the paper will be devoted to the Beurling, Cowling-Price and Gelfand-Shilov theorems. Finally in Section 4 we will prove the Donoho-Stark theorem.

2 Hardy Uncertainty Principle

Hardy's classical theorem for the Fourier transform [6, 16] says, that if $|f(y)| \le Ce^{-ay^2}$ and $|(F_c f)(x)| \le Ce^{-\frac{x^2}{4a}}$, $a > 0$, then $f(y)$ is a multiple of e^{-ay^2}. Here $C > 0$ is a universal constant, which is different in distinct places and

$$(F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \cos(xy)\, dy, \tag{2.1}$$

is the cosine Fourier transform.

Let us suppose that transformation (1.1) admits the following series expansion with respect to an index of the modified Bessel functions

$$K_{ix}[f] = \frac{C}{\cosh(\pi x/2)} \sum_{n=0}^\infty \alpha_n \left[K_{i(\frac{x}{2}+n)}\left(\frac{a}{2}\right) + K_{i(\frac{x}{2}-n)}\left(\frac{a}{2}\right)\right], \; a > 0, \tag{2.2}$$

where $\sum_{n=0}^\infty |\alpha_n| < \infty$. We have

Theorem 1. *Let $K_{ix}[f]$ satisfy (2.2) and $|f(y)| \le Ce^{-\frac{y^2}{4a}}$. Then $f(y)$ is a multiple of $e^{-\frac{y^2}{4a}}$.*

Proof. Taking (1.9), we find

$$K_{i(\frac{x}{2}+n)}\left(\frac{a}{2}\right) = \int_0^\infty e^{-\frac{a}{2}\cosh u} \cos\left(\frac{x}{2}+n\right) u\, du. \tag{2.3}$$

Hence $\left|K_{i(\frac{x}{2}+n)}\left(\frac{a}{2}\right)\right| \le K_0\left(\frac{a}{2}\right)$, and clearly series (2.2) is uniformly convergent on $\mathbb{R}_+$. Moreover, we can calculate the cosine Fourier transform of the function $\cosh(\pi x/2) K_{ix}[f]$ by changing the order of integration and summation. Indeed, using (2.3) we obtain

$$\begin{aligned} F_c(\cosh\left(\frac{\pi t}{2}\right) K_{it}[f])(x) &= C \sum_{n=0}^\infty \alpha_n \int_0^\infty \left[K_{i(\frac{t}{2}+n)}\left(\frac{a}{2}\right) + K_{i(\frac{t}{2}-n)}\left(\frac{a}{2}\right)\right] \cos(xt)\, dt \\ &= \frac{C}{2} \sum_{n=0}^\infty \alpha_n \int_{-\infty}^\infty \left[K_{i(\frac{t}{2}-n)}\left(\frac{a}{2}\right) + K_{i(\frac{t}{2}+n)}\left(\frac{a}{2}\right)\right] e^{ixt}\, dt \\ &= 2\pi C e^{-\frac{a}{2}\cosh(2x)} \sum_{n=0}^\infty \alpha_n \cos(2xn). \end{aligned}$$

Therefore

$$|F_c(\cosh(\pi t/2)K_{it}[f])(x)| \le Ce^{-\frac{a}{2}\cosh(2x)} = O\left(e^{-a\sinh^2 x}\right).$$

Further, it is easily seen under conditions of the theorem and asymptotic behavior of the modified Bessel function (1.5), (1.7), that $f \in L_1(\mathbb{R}_+; K_0(y)\,dy)$. Moreover, by virtue of the asymptotic formula with respect to an index (1.8), we verify that $K_{ix}[f] \in L_1(\mathbb{R}_+; x\cosh\frac{\pi x}{2}\,dx)$. Consequently, calling (1.3), (1.4) we find

$$yf(y) = \frac{4}{\pi^2}\int_0^\infty x\sinh\frac{\pi x}{2}\cosh\frac{\pi x}{2}K_{ix}(y)K_{ix}[f]\,dx. \tag{2.4}$$

However, since $\sinh\frac{\pi x}{2}K_{ix}(y)$ is bounded for any $y > 0$ (see (1.8)), we take the representation (1.10) and substitute it in (2.4). As a result we obtain

$$yf(y) = \frac{4}{\pi^2}\lim_{N\to\infty}\int_0^N x\cosh\frac{\pi x}{2}K_{ix}[f]\int_0^\infty \sin(y\sinh u)\sin(xu)\,du\,dx$$

$$= \frac{4}{\pi^2}\lim_{N\to\infty}\int_0^N x\cosh\frac{\pi x}{2}K_{ix}[f]\int_0^\infty \sin(yv)\sin(x\log(v+\sqrt{v^2+1}))\frac{dv\,dx}{\sqrt{v^2+1}}.$$

Via Abel's test we observe, that the latter integral is uniformly convergent with respect to $x \in [0, N]$. Thus inverting the order of integration, we come out with

$$yf(y)=\frac{4}{\pi^2}\lim_{N\to\infty}\int_0^\infty \frac{\sin(yv)}{\sqrt{v^2+1}}\int_0^N x\cosh\frac{\pi x}{2}K_{ix}[f]\sin(x\log(v+\sqrt{v^2+1}))\,dx\,dv. \tag{2.5}$$

Moreover, the integrability condition $K_{ix}[f] \in L_1(\mathbb{R}_+; x\cosh\frac{\pi x}{2}\,dx)$ and the Abel test allow us to pass to the limit under the integral sign in (2.5). Hence returning to the old variables we get

$$yf(y)= \frac{4}{\pi^2}\int_0^\infty \sin(y\sinh u)\int_0^\infty x\cosh\frac{\pi x}{2}K_{ix}[f]\sin(ux)\,dx\,du$$

$$= -\frac{4}{\pi^2}\int_0^\infty \sin(y\sinh u)\frac{d}{du}\int_0^\infty \cosh\frac{\pi x}{2}K_{ix}[f]\cos(ux)\,dx\,du. \tag{2.6}$$

We note, that the differentiation under the integral sign in (2.6) is motivated by the uniform convergence by $u \in \mathbb{R}_+$ of the latter integral with respect to x. Hence, integrating by parts in (2.6) and eliminating the outer terms owing to the Riemann-Lebesgue lemma we take into account (2.1) to derive the representation

$$f(y) = \frac{2\sqrt{2}}{\pi\sqrt{\pi}}\int_0^\infty \cos(y\sinh u)\cosh u F_c(\cosh(\pi t/2)K_{it}[f])(u)du.$$

Appealing to the above estimates and the value of an elementary integral, we find for any complex variable $z,\ |z| = r$,

$$|f(z)| < C\int_0^\infty \cosh(r\sinh u)\cosh u|F_c(\cosh(\pi t/2)K_{it}[f])(u)|\,du$$
$$\le C\int_0^\infty \cosh(r\sinh u)\ e^{-a\sinh^2 u}\cosh u\,du$$
$$= C\int_0^\infty \cosh(rt)e^{-at^2}\,dt = Ce^{\frac{r^2}{4a}}.$$

Thus $f(\sqrt{z})$ is an entire function, which is $O(e^{\frac{|z|}{4a}})$ for all $z \in \mathbb{C}$ and $f(\sqrt{y}) = O(e^{-\frac{y}{4a}}), y \in \mathbb{R}_+$. Therefore according to [16], Theorem 128, $f(y) = Ce^{-\frac{y^2}{4a}}$. Theorem 1 is proved. $\square$

Corollary 1. Under conditions of Theorem 1

$$K_{ix}[f] = C\ \mathrm{sech}(\pi x/2)K_{ix/2}\left(\frac{a}{2}\right) = O(e^{-\frac{3\pi}{4}x}), \quad x \to +\infty.$$

Proof. Indeed, substituting the value $f(y) = Ce^{-\frac{y^2}{4a}}$ into (1.1) we just call the relation (2.16.8.3) from [12], Vol. 2, to get the result. The required asymptotic behavior at infinity immediately follows from (1.8). Corollary 1 is proved. $\square$

Remark 1. As we see, $K_{ix}[f]$ from the corollary admits the representation (2.2) with $\alpha_0 \neq 0,\ \alpha_n = 0,\ n = 1, 2 \ldots$.

As a consequence we are ready to state an analog of the Hardy uncertainty principle for the Kontorovich-Lebedev transformation (1.1).

Corollary 2. Let $|f(y)| \le Ce^{-by^2},\ b > \frac{1}{4a}$. Then $f(y) = 0$.

This principle can be formulated in terms of composition $F_c(\cosh(\pi t/2)K_{it}[f])$.

Corollary 3. One cannot have both

$$|f(y)| \le Ce^{-ay^2},\ a > 0,\ \ |F_c(\cosh(\pi t/2)K_{it}[f])(x)| \le Ce^{-b\sinh^2 x},\ b > 0,$$

where $ab > \frac{1}{4}$ unless $f(y) = 0$.

As a consequence of Theorem 1 and Corollary 1 we get

Corollary 4. Let $|f(y)| \le Ce^{-ay^2},\ a > 0$ and $|F_c(\cosh(\pi t/2)K_{it}[f])(x)| \le Ce^{-b\sinh^2 x},\ b > 0$, where $0 < ab \le \dfrac{1}{4}$. If $|K_{ix}[f]| \le Ce^{-cx},\ x > 0$, $c > \dfrac{3\pi}{4}$, then $f(y) = 0$.

3 Beurling, Cowling-Price and Gelfand-Shilov Theorems

The Beurling condition related to the cosine Fourier transform (2.1) says (cf. [7]), that if $f \in L_1(\mathbb{R}_+; dy)$ and

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |f(y)(F_c f)(x)| e^{xy}\, dx\, dy < \infty, \tag{3.1}$$

then $f = 0$. Here we will prove an analog of the Beurling theorem for the Kontorovich-Lebedev transformation (1.1).

Theorem 2. *Let $f \in L_1(\mathbb{R}_+; K_0(y)\, dy)$ and*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |f(y) K_{ix}[f]|\, K_x(y)\, dx\, dy < \infty, \tag{3.2}$$

then $f = 0$.

Proof. We can assume that $f(y) \neq 0$ on a set of the positive measure $K_0(y)dy$, for otherwise there is nothing to prove. Since representation (1.9) for the modified Bessel function yields the inequality $K_x(y) \geq K_0(y)$, condition (3.2) implies

$$\infty > \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |f(y) K_{ix}[f]|\, K_x(y)\, dx\, dy \geq \int_{\mathbb{R}_+} |f(y)| K_0(y)\, dy \int_{\mathbb{R}_+} |K_{ix}[f]|\, dx.$$

Therefore $K_{ix}[f] \in L_1(\mathbb{R}_+; dx)$. The latter condition guarantees the existence of the cosine Fourier transform of $K_{ix}[f]$. We will show that

$$(F_c K_{it}[f])(\lambda) = \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-y\cosh\lambda} f(y)\, dy. \tag{3.3}$$

Indeed, denoting by $h(\lambda)$ the right-hand side of (3.3) we find

$$\int_{\mathbb{R}_+} |h(\lambda)|\, d\lambda \leq \sqrt{\frac{\pi}{2}} \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} e^{-y\cosh\lambda} |f(y)|\, dy\, d\lambda = \sqrt{\frac{\pi}{2}} \int_{\mathbb{R}_+} |f(y)| K_0(y)\, dy < \infty.$$

So $h \in L_1(\mathbb{R}_+; d\lambda)$ and $(F_c h)(x)$ can be now easily calculated by using (1.9) and Fubini's theorem. Thus we obtain

$$(F_c h)(x) = \int_0^\infty \cos x\lambda \int_0^\infty e^{-y\cosh\lambda} f(y)\, dy\, d\lambda = \int_0^\infty K_{ix}(y) f(y)\, dy K_{ix}[f].$$

Since $K_{ix}[f] \in L_1(\mathbb{R}_+; dx)$ the inversion theorem for the cosine Fourier transform gives $(F_c K_{it}[f])(\lambda) = h(\lambda)$ and we establish equality (3.3).

Let us verify the Beurling condition (3.1) for $K_{ix}[f], (F_c K_{it}[f])$. We have

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |K_{ix}[f](F_c K_{it}[f])(\lambda)| e^{x\lambda}\, dx\, d\lambda < \sqrt{2\pi} \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |K_{ix}[f]| \cosh x\lambda$$
$$\times \int_0^\infty e^{-y\cosh\lambda} |f(y)|\, dy\, dx\, d\lambda\, \sqrt{2\pi} \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |f(y) K_{ix}[f]|\, K_x(y)\, dx\, dy < \infty.$$

Thus $K_{ix}[f] = 0$. Combining with (3.3) the latter condition yields

$$\int_0^\infty e^{-y\cosh\lambda} f(y)\, dy = 0,\ \ \lambda \in \mathbb{R}_+ \tag{3.4}$$

for any $f \in L_1(\mathbb{R}_+; K_0(y)\, dy)$. We will show that in this case $f = 0$. In fact, choosing any $\lambda_0 > 1$ we treat the left-hand side of equality (3.4) as the Laplace integral $(Lf)(\cosh\lambda)$, where

$$(Lf)(z) = \int_0^\infty e^{-yz} f(y)\, dy,$$

which is zero via (3.4) at least at the countable set of points satisfying the condition $\cosh\lambda_n = \lambda_0 + jn,\ j > 0, n = 1, 2, \dots$. Moreover, since (see (1.5), (1.7))

$$\int_0^\infty e^{-y\cosh\lambda_n} |f(y)|\, dy < \infty,\ \ n = 1, 2, \dots\ ,$$

then by virtue of [3], Chapter I we get that $f(y) = 0$ almost for all $y \in \mathbb{R}_+$, i.e. $f = 0$ in the Lebesgue sense. Theorem 2 is proved. □

Let us prove an analog of the Gelfand-Shilov uncertainty principle for the transformation (1.1). Indeed, it was shown in [5] that if

$$\int_{\mathbb{R}_+} |f(y)| e^{(ay)^p/p}\, dy < \infty, \quad \int_{\mathbb{R}_+} |(F_c f)(x)| e^{(bx)^q/q}\, dx < \infty,$$

with $1 < p, q < \infty,\ p^{-1} + q^{-1} = 1$ and $ab > 1/4$, then $f = 0$. We have accordingly

Theorem 3. *Let* $1 < p, q < \infty,\ p^{-1} + q^{-1} = 1,\ [q]$ *be an integer part of* q *and*

$$\int_{\mathbb{R}_+} |f(y^2)| e^{\frac{(2([q]+1))!}{4y^2}}\, dy < \infty, \quad \int_{\mathbb{R}_+} |K_{ix}[f]|\, e^{x^p/p}\, dx < \infty. \tag{3.5}$$

Then $f = 0$.

Proof. By using the Young inequality $xy \le \dfrac{x^p}{p} + \dfrac{y^q}{q}$ and representation (1.9) for the modified Bessel function we derive

$$\begin{aligned} K_x(y) &= \int_0^\infty e^{-y\cosh u} \cosh xu du \le \int_0^\infty e^{-y\cosh u + xu}\, du \\ &\le e^{x^p/p} \int_0^\infty e^{-y\cosh u + \frac{u^q}{q}}\, du = e^{x^p/p} \left(\int_0^1 + \int_1^\infty \right) e^{-y\cosh u + \frac{u^q}{q}}\, du. \end{aligned} \tag{3.6}$$

Meanwhile,

$$\int_0^1 e^{-y\cosh u + u^q/q} du < e \int_0^1 e^{-y\cosh u} du < eK_0(y),$$

$$\int_1^\infty e^{-y\cosh u + u^q/q} du < ([q]+1) \int_1^\infty e^{-y\cosh u + u^{[q]+1}} u^{[q]}\, du.$$

Therefore an elementary inequality $\cosh u > \frac{u^{2([q]+1)}}{(2([q]+1))!}$ gives the following estimation of the latter integral

$$\int_1^\infty e^{-y\cosh u + \frac{u^q}{q}} du < ([q]+1)\int_1^\infty e^{-y\cosh u + u^{[q]+1}} u^{[q]}\, du$$
$$< ([q]+1)\int_1^\infty e^{-\frac{yu^{2([q]+1)}}{(2([q]+1))!} + u^{[q]+1}} u^{[q]} du$$
$$= \int_1^\infty e^{-\frac{yv^2}{(2([q]+1))!} + v} dv < \frac{C}{\sqrt{y}} e^{(2([q]+1))!/(4y)}.$$

Combining with (3.6) and taking into account the asymptotic formulas (1.5), (1.7), we obtain the estimate

$$e^{-x^p/p} K_x(y) < eK_0(y) + \frac{C}{\sqrt{y}} e^{(2([q]+1))!/(4y)} < \frac{C}{\sqrt{y}} e^{(2([q]+1))!/(4y)}.$$

Consequently, with conditions (3.2), (3.5) and since via (3.5) we have that $f \in L_1(\mathbb{R}_+; K_0(y)\, dy)$, it yields

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |f(y)K_{ix}[f]|\, K_x(y)\, dx\, dy$$
$$< C\int_{\mathbb{R}_+} |K_{ix}[f]|\, e^{x^p/p}\, dx \int_{\mathbb{R}_+} |f(y)| e^{(2([q]+1))!/(4y)} \frac{dy}{\sqrt{y}}$$
$$= C\int_{\mathbb{R}_+} |K_{ix}[f]|\, e^{x^p/p}\, dx \int_{\mathbb{R}_+} |f(y^2)| e^{(2([q]+1))!/4y^2}\, dy < \infty.$$

Applying Theorem 2 we get that $f = 0$. Theorem 3 is proved. □

Finally in this section we establish the Cowling-Price theorem for the Kontorovich-Lebedev transform (1.1). This will be an analog of the following result for the Fourier transform (2.1) (cf. [11]: if $1 \le p, q < \infty$ and

$$\left\| e^{ax^2} f(x)\right\|_{L_p(\mathbb{R}_+)} + \left\| e^{b\lambda^2} (F_c f)(\lambda)\right\|_{L_q(\mathbb{R}_+)} < \infty$$

with $ab > 1/4$, then $f = 0$. We have

Theorem 4. *If*

$$\left\| e^{ax^2} K_{ix}[f]\right\|_{L_p(\mathbb{R}_+)} < \infty, \quad \left\| e^{6b^2/y^2} f(y^2)\right\|_{L_1(\mathbb{R}_+)} < \infty,$$

where $p \in [1, \infty)$ *and* $ab > 1/4$, *then* $f = 0$.

Proof. Indeed, choosing a_0, b_0 such that $0 < a_0 < a\ 0 < b_0 < b,\ a_0 b_0 > 1/4$ we easily find, that

$$a_0 x^2 + b_0 y^2 \ge 2\sqrt{a_0 b_0}\, xy \ge xy.$$

Furthermore, with the Hölder inequality it gives

$$\int_{\mathbb{R}_+} |K_{ix}[f]|\, e^{a_0 x^2}\, dx \le \left\| e^{ax^2} K_{ix}[f] \right\|_{L_p(\mathbb{R}_+)} \left\| e^{-(a-a_0)x^2} \right\|_{L_{p'}(\mathbb{R}_+)} < \infty,$$

where p' is the conjugate exponent ($p^{-1} + p'^{-1} = 1$). Taking (3.2) we deduce similar to (3.6)

$$\begin{aligned} &\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} |f(y) K_{ix}[f]|\, K_x(y)\, dx\, dy \\ &\quad < \int_{\mathbb{R}_+} |K_{ix}[f]|\, e^{a_0 x^2}\, dx \int_{\mathbb{R}_+} |f(y)| \int_0^\infty e^{-y\cosh u + b_0 u^2}\, du\, dy \\ &\quad < C \left\| e^{ax^2} K_{ix}[f] \right\|_{L_p(\mathbb{R}_+)} \int_{\mathbb{R}_+} |f(y)| \left(\int_0^1 + \int_1^\infty \right) e^{-y\cosh u + b_0 u^2}\, du\, dy. \end{aligned}$$

But

$$\begin{aligned} \left(\int_0^1 + \int_1^\infty \right) e^{-y\cosh u + b_0 u^2} du &< C K_0(y) + 2 \int_1^\infty e^{-y\frac{u^4}{4!} + bu^2} u\, du \\ &= C K_0(y) + \int_1^\infty e^{-y\frac{v^2}{4!} + bv} dv < C \frac{e^{6b^2/y}}{\sqrt{y}}. \end{aligned}$$

Hence as in Theorem 3

$$\begin{aligned} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} |f(y) K_{ix}[f]|\, K_x(y)\, dx\, dy &< C \left\| e^{ax^2} K_{ix}[f] \right\|_{L_p(\mathbb{R}_+)} \int_{\mathbb{R}_+} |f(y)| \frac{e^{6b^2/y}}{\sqrt{y}}\, dy \\ &= C \left\| e^{ax^2} K_{ix}[f] \right\|_{L_p(\mathbb{R}_+)} \left\| e^{6b^2/y^2} f(y^2) \right\|_{L_1(\mathbb{R}_+)} < \infty. \end{aligned}$$

Thus the Beurling type condition (3.2) holds, and by virtue of Theorem 2 $f = 0$. Theorem 4 is proved. □

4 Donoho-Stark Theorem

It is shown in [19], when $f \in L_2(\mathbb{R}_+; y\, dy)$, then $K_{ix}[f] \in L_2(\mathbb{R}_+; x \sinh \pi x\, dx)$ and vice versa. Moreover, by virtue of (1.2)

$$\|K_{ix}[f]\|_{L_2(\mathbb{R}_+; x \sinh \pi x\, dx)} = \frac{\pi}{\sqrt{2}} \|f\|_{L_2(\mathbb{R}_+; y\, dy)}$$

and the Kontorovich-Lebedev integrals (1.1), (1.3) can be interpreted accordingly in the mean convergence sense with respect to the related norm

$$\begin{aligned} K_{ix}[f] &\equiv g(x) = \text{l.i.m.}_{N\to\infty} \int_{1/N}^N K_{ix}(y) f(y)\, dy, \\ f(y) &= \frac{2}{\pi^2} \text{l.i.m.}_{N\to\infty} \int_0^N x \sinh \pi x \frac{K_{ix}(y)}{y} K_{ix}[f]\, dx. \end{aligned} \tag{4.1}$$

Let $\mathbb{X} = [0, X]$, $\mathbb{Y} = [1/Y, Y]$ the Lebesgue measurable sets and $|\mathbb{X}|$, $|\mathbb{Y}|$ be their Lebesgue measures. Denoting by $P_{\mathbb{X}}$ the operator

$$(P_{\mathbb{X}}g)(x) = \begin{cases} g(x), & \text{if } x \in \mathbb{X}, \\ 0, & \text{if } x \notin \mathbb{X}, \end{cases}$$

we have

$$||g - P_{\mathbb{X}}g||_{L_2(\mathbb{R}_+; x \sinh \pi x\, dx)} \leq \varepsilon_{\mathbb{X}},$$

and this means that g is $\varepsilon_{\mathbb{X}}$-concentrated on the set $\mathbb{X}$. Plainly $||P_{\mathbb{X}}|| = 1$. Another auxiliary operator is given by the formula

$$(Q_{\mathbb{Y}}g)(x) = \int_{\mathbb{Y}} K_{ix}(y) f(y)\, dy,$$

where f is the reciprocal inverse Kontorovich-Lebedev transform (4.1). If $h = Q_{\mathbb{Y}}g$ the transform (4.1) $\hat{h}(y)$ is equal to

$$\hat{h}(y) = \begin{cases} f(y), & \text{if } y \in \mathbb{Y}, \\ 0, & \text{if } y \notin \mathbb{Y}. \end{cases}$$

Meanwhile by Parseval's equality (1.2) we find

$$\left\| f - \hat{h} \right\|_{L_2(\mathbb{R}_+; y\, dy)} = \frac{\sqrt{2}}{\pi} \|g - Q_{\mathbb{Y}}g\|_{L_2(\mathbb{R}_+; x \sinh \pi x\, dx)}, \tag{4.2}$$

and f is ε-concentrated on $\mathbb{Y}$ if, and only if, $\|g - Q_{\mathbb{Y}}g\|_{L_2(\mathbb{R}_+; x \sinh \pi x\, dx)} \leq \varepsilon_{\mathbb{Y}}$. Moreover, we can show that $\|Q_{\mathbb{Y}}\| = 1$.

Now we are ready to prove the following analog of the Donoho-Stark uncertainty principle (cf. [8]).

Theorem 5. *Let g is $\varepsilon_{\mathbb{X}}$-concentrated on $\mathbb{X} = [0, X]$ and its Kontorovich-Lebedev reciprocity f is $\varepsilon_{\mathbb{Y}}$-concentrated on $\mathbb{Y} = [1/Y, Y]$. Then*

$$|\mathbb{X}|^{3/2}\ |\mathbb{Y}| \geq \frac{\pi^{7/4}\sqrt{24}}{\Gamma^2(1/4)} (1 - (\varepsilon_{\mathbb{X}}^2 + \varepsilon_{\mathbb{Y}}^2)^{1/2})^2, \tag{4.3}$$

where $\Gamma(z)$ is Euler's gamma-function.

Proof. Without loss of generality we suppose that $Y > 1$. Since g is $\varepsilon_{\mathbb{X}}$-concentrated on $\mathbb{X}$ integral (1.3) exists as a Lebesgue integral and is uniformly convergent with respect to $y \in \mathbb{Y}$. Hence we calculate the following composition of operators $(P_{\mathbb{X}}Q_{\mathbb{Y}}g)(x)$. Indeed, we derive

$$\begin{aligned}(P_{\mathbb{X}}Q_{\mathbb{Y}}g)(x) &= \frac{2}{\pi^2} P_{\mathbb{X}} \int_{\mathbb{Y}} \frac{K_{ix}(y)}{y} \int_0^\infty t \sinh \pi t K_{it}(y) g(t)\, dt\, dy \\ &= \frac{2}{\pi^2} P_{\mathbb{X}} \int_0^\infty t \sinh \pi t\ g(t) \int_{\mathbb{Y}} K_{ix}(y) K_{it}(y) \frac{dy}{y}\, dt = \int_0^\infty \mathcal{K}(x,t) g(t)\, dt,\end{aligned}$$

where

$$\mathcal{K}(x,t) = \begin{cases} \frac{2}{\pi^2} t \sinh \pi t \int_{\mathbb{Y}} K_{ix}(y) K_{it}(y) \frac{dy}{y}, & \text{if } x < X, \\ 0, & \text{if } x \geq X. \end{cases}$$

Further,

$$\|P_{\mathbb{X}} Q_{\mathbb{Y}} g\|_{L_2(\mathbb{R}_+; x \sinh \pi x\, dx)} \leq \|P_{\mathbb{X}} Q_{\mathbb{Y}}\| \, \|g\|_{L_2(\mathbb{R}_+; x \sinh \pi x\, dx)}$$

and the norm of composition $P_{\mathbb{X}} Q_{\mathbb{Y}}$ does not exceed its Hilbert-Schmidt norm, which is equal to

$$\left(\int_0^\infty \int_0^\infty |\mathcal{K}(x,t)|^2 \frac{x \sinh \pi x}{t \sinh \pi t} \, dt \, dx \right)^{1/2}.$$

Therefore,

$$\begin{aligned} ||P_{\mathbb{X}} Q_{\mathbb{Y}}||^2_{L_2(\mathbb{R}_+; x \sinh \pi x\, dx)} &\leq \int_0^\infty \int_0^\infty |\mathcal{K}(x,t)|^2 \frac{x \sinh \pi x}{t \sinh \pi t} \, dt \, dx \\ &= \int_0^X \int_0^\infty |\mathcal{K}(x,t)|^2 \frac{x \sinh \pi x}{t \sinh \pi t} \, dt \, dx. \end{aligned} \tag{4.4}$$

But the inner integral with respect to t in (4.4) can be calculated by the Parseval equality (1.2), regarding $\frac{\mathcal{K}(x,t)}{t \sinh \pi t}$ as the Kontorovich-Lebedev transform (1.1) of

$$\varphi(y) = \begin{cases} \frac{2}{\pi^2} \frac{K_{ix}(y)}{y}, & \text{if } y \in \mathbb{Y}, \\ 0, & \text{if } y \notin \mathbb{Y}. \end{cases}$$

Consequently,

$$\int_0^\infty |\mathcal{K}(x,t)|^2 \frac{dt}{t \sinh \pi t} = \frac{2}{\pi^2} \int_{\mathbb{Y}} K_{ix}^2(y) \frac{dy}{y}$$

and we come out with

$$\|P_{\mathbb{X}} Q_{\mathbb{Y}}\|^2_{L_2(\mathbb{R}_+; x \sinh \pi x\, dx)} \leq \frac{2}{\pi^2} \int_{\mathbb{X}} \int_{\mathbb{Y}} x \sinh \pi x K_{ix}^2(y) \frac{dy}{y} dx. \tag{4.5}$$

Let us estimate the right-hand side of (4.5). Applying twice the Schwarz inequality we obtain

$$\begin{aligned} \frac{2}{\pi^2} \int_{\mathbb{X}} \int_{\mathbb{Y}} x \sinh \pi x K_{ix}^2(y) \frac{dy}{y} dx &\leq \frac{2}{\pi^2} \left(Y - \frac{1}{Y} \right)^{1/2} \int_{\mathbb{X}} x \sinh \pi x \left(\int_{\mathbb{Y}} K_{ix}^4(y) \, dy \right)^{1/2} dx \\ &\leq \frac{2}{\pi^2 \sqrt{3}} |\mathbb{X}|^{3/2} \sqrt{|\mathbb{Y}|} \left(\int_{\mathbb{X}} \int_{\mathbb{Y}} \sinh^2 \pi x K_{ix}^4(y) \, dy \, dx \right)^{1/2}. \end{aligned}$$

On the other hand by relation (2.16.52.17) from [12] Vol. 2, and the Parseval equality for the sine Fourier transform we find

$$\int_0^\infty \sinh^2 \pi x K_{ix}^4(y) \, dx = \frac{\pi^3}{8} \int_0^\infty J_0^2 \left(2y \sinh(u/2) \right) du,$$

where $J_0(z)$ is the Bessel function of the first kind. Consequently, employing relation (2.12.31.2) from [12] Vol. 2, and the Hölder inequality we get

$$\frac{2}{\pi^2\sqrt{3}}|\mathbb{X}|^{3/2}\sqrt{|\mathbb{Y}|}\left(\int_{\mathbb{X}}\int_{\mathbb{Y}}\sinh^2\pi x K_{ix}^4(y)\,dy\,dx\right)^{1/2}$$

$$\leq \frac{1}{\sqrt{6\pi}}|\mathbb{X}|^{3/2}\sqrt{|\mathbb{Y}|}\left(\int_{\mathbb{Y}}\int_0^\infty J_0^2\left(2y\sinh(u/2)\right)\,du\,dy\right)^{1/2}$$

$$= \frac{1}{\sqrt{3\pi}}|\mathbb{X}|^{3/2}\sqrt{|\mathbb{Y}|}\left(\int_{\mathbb{Y}}\int_0^\infty J_0^2\left(v\right)\frac{dv\,dy}{\sqrt{v^2+4y^2}}\right)^{1/2}$$

$$\leq \frac{|\mathbb{X}|^{3/2}\sqrt{|\mathbb{Y}|}}{\sqrt{6\pi}}\left(\int_{\mathbb{Y}}\frac{dy}{\sqrt{y}}\int_0^\infty J_0^2\left(v\right)\frac{dv}{\sqrt{v}}\right)^{1/2}$$

$$= \frac{\Gamma^2(1/4)}{2\pi^{7/4}\sqrt{6}}|\mathbb{X}|^{3/2}\sqrt{|\mathbb{Y}|}\left(\int_{\mathbb{Y}}\frac{dy}{\sqrt{y}}\right)^{1/2}$$

$$\leq \frac{\Gamma^2(1/4)}{2\pi^{7/4}\sqrt{6}}|\mathbb{X}|^{3/2}|\mathbb{Y}|^{7/8}\left(\int_{\mathbb{Y}}\frac{dy}{y^2}\right)^{1/8} = \frac{\Gamma^2(1/4)}{2\pi^{7/4}\sqrt{6}}|\mathbb{X}|^{3/2}|\mathbb{Y}|.$$

Thus combining with (4.4) we derive finally the inequality

$$\|P_{\mathbb{X}}Q_{\mathbb{Y}}\|_{L_2(\mathbb{R}_+;x\sinh\pi x\,dx)} \leq \frac{\Gamma(1/4)}{\sqrt{2\sqrt{6}}\pi^{7/8}}|\mathbb{X}|^{3/4}|\mathbb{Y}|^{1/2}.$$

But $\|P_{\mathbb{X}}Q_{\mathbb{Y}}\|_{L_2(\mathbb{R}_+;x\sinh\pi x\,dx)} < 1$, and therefore, $I - P_{\mathbb{X}}Q_{\mathbb{Y}}$ is invertible in $L_2(\mathbb{R}_+;x\sinh\pi x\,dx)$ when $|\mathbb{X}|^{3/4}|\mathbb{Y}|^{1/2} < \frac{\sqrt{2\sqrt{6}}\pi^{7/8}}{\Gamma(1/4)}$. Moreover,

$$\|(I-P_{\mathbb{X}}Q_{\mathbb{Y}})^{-1}\| \leq \sum_{n=0}^\infty \|P_{\mathbb{X}}Q_{\mathbb{Y}}\|^n \leq \sum_{n=0}^\infty\left[\frac{\Gamma(1/4)}{\sqrt{2\sqrt{6}}\pi^{7/8}}|\mathbb{X}|^{3/4}|\mathbb{Y}|^{1/2}\right]^n$$

$$= \frac{\sqrt{2\sqrt{6}}\pi^{7/8}}{\sqrt{2\sqrt{6}}\pi^{7/8} - \Gamma(1/4)|\mathbb{X}|^{3/4}|\mathbb{Y}|^{1/2}}.$$

However,

$$I = P_{\mathbb{X}} + P_{\mathbb{R}_+\setminus\mathbb{X}}P_{\mathbb{X}}Q_{\mathbb{Y}} + P_{\mathbb{X}}Q_{\mathbb{R}_+\setminus\mathbb{Y}} + P_{\mathbb{R}_+\setminus\mathbb{X}}$$

and the orthogonality $P_{\mathbb{X}}$ and $P_{\mathbb{R}_+\setminus\mathbb{X}}$ gives

$$\|P_{\mathbb{X}}Q_{\mathbb{R}_+\setminus\mathbb{Y}}g\|^2_{L_2(\mathbb{R}_+;x\sinh\pi x\,dx)} + \|P_{\mathbb{R}_+\setminus\mathbb{X}}g\|^2_{L_2(\mathbb{R}_+;x\sinh\pi x\,dx)}$$

$$= \|P_{\mathbb{X}}Q_{\mathbb{R}_+\setminus\mathbb{Y}}g + P_{\mathbb{R}_+\setminus\mathbb{X}}g\|^2_{L_2(\mathbb{R}_+;x\sinh\pi x\,dx)}.$$

Taking into account that $\|P_{\mathbb{X}}\| = 1$ we find

$$\|g\|^2_{L_2(\mathbb{R}_+;x\sinh\pi x dx)} \leq \|(I-P_{\mathbb{X}}Q_{\mathbb{Y}})^{-1}\|^2\|(I-P_{\mathbb{X}}Q_{\mathbb{Y}})g\|^2_{L_2(\mathbb{R}_+;x\sinh\pi x\,dx)}$$

$$\leq \left(\frac{\sqrt{2\sqrt{6}}\pi^{7/8}}{\sqrt{2\sqrt{6}}\pi^{7/8} - \Gamma(1/4)|\mathbb{X}|^{3/4}|\mathbb{Y}|^{1/2}} \right)^2 \big[\|P_{\mathbb{X}} Q_{\mathbb{R}_+\setminus\mathbb{Y}} g\|^2_{L_2(\mathbb{R}_+; x\sinh \pi x\, dx)}$$

$$+ \|P_{\mathbb{R}_+\setminus\mathbb{X}} g\|^2_{L_2(\mathbb{R}_+; x\sinh \pi x\, dx)} \big] \leq \left(\frac{\sqrt{2\sqrt{6}}\pi^{7/8}}{\sqrt{2\sqrt{6}}\pi^{7/8} - \Gamma(1/4)|\mathbb{X}|^{3/4}|\mathbb{Y}|^{1/2}} \right)^2$$

$$\times \left[\|Q_{\mathbb{R}_+\setminus\mathbb{Y}} g\|^2_{L_2(\mathbb{R}_+; x\sinh \pi x\, dx)} + \|P_{\mathbb{R}_+\setminus\mathbb{X}} g\|^2_{L_2(\mathbb{R}_+; x\sinh \pi x\, dx)} \right].$$

Now since g is $\varepsilon_{\mathbb{X}}$-concentrated then $\|P_{\mathbb{R}_+\setminus\mathbb{X}} g\|_{L_2(\mathbb{R}_+; x\sinh \pi x dx)} \leq \varepsilon_{\mathbb{X}}$. Furthermore, since f is $\varepsilon_{\mathbb{Y}}$-concentrated then owing to (4.2) we have the estimate $\|Q_{\mathbb{R}_+\setminus\mathbb{Y}} g\|_{L_2(\mathbb{R}_+; x\sinh \pi x\, dx)} \leq \varepsilon_{\mathbb{Y}}$. Therefore considering g of unit norm we arrive at the inequality

$$1 \leq \left(\frac{\sqrt{2\sqrt{6}}\pi^{7/8}}{\sqrt{2\sqrt{6}}\pi^{7/8} - \Gamma(1/4)|\mathbb{X}|^{3/4}|\mathbb{Y}|^{1/2}} \right)^2 (\varepsilon^2_{\mathbb{X}} + \varepsilon^2_{\mathbb{Y}}),$$

which is equivalent to (4.3). Theorem 5 is proved. □

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