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A Computational Method for Nonlinear $2m$-th Order Boundary Value Problems*

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Abstract. In this paper, two point boundary value problems of $2m$th-order nonlinear differential equations are considered. The existence of the solution and a new iterative algorithm which is large-range convergent are proposed for the problems in reproducing kernel space. The advantage of the approach must lie in the fact that, on the one hand, for the arbitrary fixed initial value function, the iterative method is convergent. On the other hand, the approximate solution and its derivatives converge uniformly to the exact solution and its derivatives, respectively. Some examples are displayed to demonstrate the computation efficiency of the method.

Keywords: boundary value problems, nonlinear differential equation, existence, reproducing kernel space.

AMS Subject Classification: 30E25; 34B16.

1 Introduction

Consider the following $2m$-th order two-point boundary value problems(BVPs)

$$y^{(2m)}(x) = f(x, y), \quad x \in (0, b) \tag{1.1}$$

subject to boundary conditions

$$y^{(2l)}(0) = \alpha_{2l}, \quad y^{(2l)}(b) = \beta_{2l}, \quad l = 0, 1, 2, \ldots, m-1, \tag{1.2}$$

or with the initial conditions given for the starting point

$$y^{(l)}(0) = \gamma_l, \quad l = 0, 1, 2, \ldots, 2m-1, \tag{1.3}$$

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where $y(x) \in W_1^{2m+1}[0,b]$ for problem (1.1) – (1.2), $y(x) \in W_2^{2m+1}[0,b]$ for the initial value problem (1.1), (1.3), for $x \in [0,b]$, $z \in (-\infty,\infty)$, $f(x,z)$ is a continuous bounded function, $z = z(x)$, $f(x,z) \in W_1^1[0,b]$; b, α_{2l}, β_{2l}, $l = 0,1,2,\ldots,m-1$ and γ_l, $l = 0,1,2\ldots,2m-1$ are constants; $W_1^{2m+1}[0,b]$, $W_2^{2m+1}[0,b]$ and $W_1^1[0,b]$ are reproducing kernel spaces.

High-order BVPs arise in many fields. For example, the narrow convecting layers bounded by stable layers which are believed to surround A-type stars may be modelled by 6th-order BVPs. When an infinite horizontal layer of fluid is heated from below and is subject to the action of rotation, instability sets in. When the instability sets in as overstability, it is modelled by 8th-order BVPs. Even higher-order BVPs can be involved when a uniform magnetic field is applied across the fluid in the same direction as gravity. Ordinary convection and overstability yield 10th-order and 12th-order BVPs, respectively. For more details about the occurrences of high-order BVPs, see [3, 4].

Many authors have investigated the BVPs of high-order because of both their mathematical importance and their potential for applications in hydrodynamic and hydromagnetic stability and so on. Agarwal [1] presented the theorems stating the conditions for the existence and uniqueness of solutions of such BVPs, while no numerical methods are contained therein. In [21], the author discussed the sufficient conditions for existence of multiple solutions of nonlinear fourth-order Emden–Fowler type equations based on the oscillation theory by Leighton and Nehari for linear fourth-order differential equations. In [8], T. Garbuza presents a special technique based on the analysis of oscillatory behaviour of linear equations to investigation of the 6th order nonlinear boundary value problem. Non-polynomial spline technique [2, 15], polynomial splines of degree six [17], generalised differential quadrature rule [13] and the spline method [16, 18] are developed for linear 4th-order, 6th-order, 8th-order and 10th-order BVPs, respectively. Using finite-difference methods [7], computational results for special nonlinear BVPs of the $2m$-th order have also been obtained. Adomian decomposition method [14, 19, 20] is applied to construct the numerical solution for nonlinear high-order BVPs.

Reproducing kernel theory has important applications in numerical analysis, differential equations, probability and statistics [5, 6, 9, 10, 11, 22]. Recently, using the reproducing kernel space method, some authors discussed nonlinear operator equations, singular linear two-point boundary value problems, singular nonlinear two-point periodic boundary value problems, nonlinear systems of boundary value problems and nonlinear partial differential equations and so on [5, 6, 9, 11, 12, 22].

In this study, the existence of the solution and a new iterative algorithm are established for the nonlinear $2m$th-order BVPs (1.1) with (1.2) or (1.3) in reproducing kernel space. The advantage of the approach must lie in the fact that, on the one hand, the iterative method is convergent for arbitrary initial value function $y_1(x)$. Therefore, we get a large-range convergence iterative method. On the other hand, the approximate solution $y_n(x)$ and the exact solution $y(x)$ satisfy $\|y_n^{(k)} - y^{(k)}\|_C \to 0, k = 0,1,2,\ldots,2m$ as $n \to \infty$.

The paper is organized as follows. In Section 2, we introduce some definitions of the reproducing kernel spaces and give the transformation of Eq.(1.1).

Section 3 provides the main results, the existence of the solution to Eq.(1.1) and a iterative method are developed for the problems in reproducing kernel space. We verify that the approximate solution converges to the exact solution uniformly. Furthermore, we obtain that the approximate solution $y_n(x)$ and the exact solution $y(x)$ satisfy $\|y_n^{(k)} - y^{(k)}\|_C \to 0, k = 1, 2, \ldots, 2m$ as $n \to \infty$. Some experiments are presented in Section 4. Finally, in Section 5 we give some conclusions.

2 Preliminaries

Let us introduce the definitions of several reproducing kernel spaces.

2.1 The reproducing kernel spaces $W_1^{2m+1}[0,b]$, $W_2^{2m+1}[0,b]$, $W_1^1[0,b]$

Definition 1. $W_1^{2m+1}[0,b] = \big\{y(x)\big|\ y, y^{(1)}, y^{(2)}, \ldots, y^{(2m)}$ are absolutely continuous real-valued functions in $[0,b]$, $y^{(2l)}(0) = 0$, $y^{(2l)}(b) = 0$, $l = 0, 1, 2 \ldots,$ $m-1$, $y^{(2m+1)} \in L^2[0,b]\big\}$.

$W_1^{2m+1}[0,b]$ is a Hilbert space, for $y, z \in W_1^{2m+1}[0,b]$, the inner product and norm in $W_1^{2m+1}[0,b]$ are given by

$$< y, z >_{W_1^{2m+1}} = \int_0^b (y^{(2m)} z^{(2m)} + y^{(2m+1)} z^{(2m+1)})\, dx, \ \|y\|_{W_1^{2m+1}} = < y, y >^{\frac{1}{2}},$$

respectively. $W_1^{2m+1}[0,b]$ is a reproducing kernel space. That is, for each fixed $x \in [0,b]$ and any $y(t) \in W_1^{2m+1}[0,b]$, there exists $R_x(t) \in W_1^{2m+1}[0,b], t \in [0,b]$ such that $< y(t), R_x(t) >_{W_1^{2m+1}} = y(x)$, the reproducing kernel $R_x(t)$ can be presented by

$$R_x(t) = \begin{cases} \sum\limits_{i=1}^{4m} a_i t^{i-1} + a_{4m+1} e^t + a_{4m+2} e^{-t}, & t \le x, \\ \sum\limits_{i=1}^{4m} b_i t^{i-1} + b_{4m+1} e^t + b_{4m+2} e^{-t}, & t > x, \end{cases} \tag{2.1}$$

where $a_i, b_i, i = 1, 2, \ldots, 4m+2$ are functions of x and they are obtained in Appendix.

Definition 2. $W_2^{2m+1}[0,b] = \big\{y(x)\big|\ y, y^{(1)}, y^{(2)}, \ldots, y^{(2m)}$ are absolutely continuous real-valued functions in $[0,b]$, $y^{(l)}(0) = 0$, $l = 0, 1, 2 \ldots, 2m-1$, $y^{(2m+1)} \in L^2[0,b]\big\}$.

$W_2^{2m+1}[0,b]$ is a Hilbert space, for $y, z \in W_2^{2m+1}[0,b]$, the inner product and norm in $W_2^{2m+1}[0,b]$ are given by

$$< y, z >_{W_2^{2m+1}} = \int_0^b (y^{(2m)} z^{(2m)} + y^{(2m+1)} z^{(2m+1)})\, dx, \ \|y\|_{W_2^{2m+1}} = < y, y >^{\frac{1}{2}},$$

respectively. $W_2^{2m+1}[0,b]$ is a reproducing kernel space. That is, for each fixed $x \in [0,b]$ and any $y(t) \in W_2^{2m+1}[0,b]$, there exist $R_x^{\{1\}}(t) \in W_2^{2m+1}[0,b], t \in$

$[0,b]$ such that $< y(t), R_x^{\{1\}}(t) >_{W_2^{2m+1}} = y(x)$, the reproducing kernel $R_x(t)$ can be presented by

$$R_x^{\{1\}}(t) = \begin{cases} \sum\limits_{i=1}^{4m} c_i t^{i-1} + c_{4m+1} e^t + c_{4m+2} e^{-t}, & t \le x, \\ \sum\limits_{i=1}^{4m} d_i t^{i-1} + d_{4m+1} e^t + d_{4m+2} e^{-t}, & t > x, \end{cases} \tag{2.2}$$

where $c_i, d_i, i = 1, 2, \ldots, 4m+2$ are functions of x, which are obtained in Appendix.

DEFINITION 3. $W_1^1[0,b] = \{y(x) \big| \ y$ is absolutely continuous real-valued function, $y' \in L^2[0,b]\}$.

$W_1^1[0,b]$ is a Hilbert space, the inner product and norm in $W_1^1[0,b]$ are given by

$$< y, z >_{W_1^1} = \int_0^b (yz + y^{(1)} z^{(1)})\, dx, \ \|y\|_{W_1^1} = < y, y >^{\frac{1}{2}},$$

respectively, where $y, z \in W_1^1[0,b]$. In [11], the authors have proved that $W_1^1[0,b]$ is a complete reproducing kernel space and its reproducing kernel is

$$R_x^{\{2\}}(t) = \frac{1}{2\sinh(b)}[\cosh(x+t-b) + \cosh(|x-t|-b)].$$

Remark 1. We describe the main results for two cases: in Case (i) we discuss Eq.(1.1) subjected to (1.2) in $W_1^{2m+1}[0,b]$; and in Case (ii) we discuss Eq.(1.1) subjected to (1.3) in $W_2^{2m+1}[0,b]$. For simplicity, we present full proofs only for Case (i), but the reader can easily verify that essentially the same proofs work for Case (ii).

2.2 Transformation

Let's consider Case (i) and define $\hat{y}(x) = y(x) + u(x)$ such that $\hat{y}^{(2l)}(0) = 0$, $\hat{y}^{(2l)}(b) = 0$, $u^{(2l)}(0) = -\alpha_{2l}$, $u^{(2l)}(b) = -\beta_{2l}$, $l = 0, 1, 2, \ldots, m-1$. We take the differential operator $T = \dfrac{d^{2m}}{dx^{2m}}$, then after homogenization of boundary conditions and denoting $\hat{y}(x)$ by $y(x)$, we put Eq.(1.1) with (1.2) into the following form:

$$\begin{cases} Ty = g(x,y), & x \in [0,b], \\ Ty = g(x,y), & x \in (0,b), \\ y^{(2l)}(b) = 0, & l = 0, 1, 2 \ldots, m-1, \end{cases} \tag{2.3}$$

where $y(x) \in W_1^{2m+1}[0,b]$, $g(x,y) = f(x, y-u) - u^{(2m)}(x)$, for $x \in [0,b], z \in (-\infty, +\infty)$, $g(x,z)$ is a continuous bounded function, $z = z(x), g(x,z) \in W_1^1[0,b]$. It is clear that $T : W_1^{2m+1}[0,b] \to W_1^1[0,b]$ is a bounded linear operator. Let $\varphi_i(x) = R_{x_i}(x)$, $\psi_i(x) = T^* \varphi_i(x)$, where $\{x_i\}_{i=1}^{\infty}$ is dense in $[0,b]$, for $y(x) \in W_1^{2m+1}[0,b]$,

$$< y(x), \varphi_i(x) >_{W_1^{2m+1}} = y(x_i),$$

T^* is the conjugate operator of T. Let us define the orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ in $W_1^{2m+1}[0,b]$ which is derived from Gram–Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad \beta_{ii} > 0, \ \ i = 1, 2, \ldots .$$

Lemma 1. *Assume $\{x_i\}_{i=1}^{\infty}$ is dense in $[0,b]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system in $W_1^{2m+1}[0,b]$ and $\psi_i(x) = T_\xi R_x(\xi)|_{\xi=x_i}$.*

Proof. One has that

$$\begin{aligned}&\psi_i(x) = (T^*\varphi_i)(x) =< (T^*\varphi_i)(\xi), \quad R_x(\xi) >_{W_2^{2m+1}} =< \varphi_i(\xi),\\ &T_\xi R_x(\xi) >_{W_1^{2m+1}} = T_\xi R_x(\xi)|_{\xi=x_i}.\end{aligned}$$

Clearly, $\psi_i(x) \in W_1^{2m+1}[0,b]$. For any function $y(x) \in W_1^{2m+1}[0,b]$, let us take $< y(x), \psi_i(x) >_{W_2^{2m+1}} = 0$, $i = 1, 2, \ldots$, which means that,

$$< y(x), T^*\varphi_i(x) >_{W_1^{2m+1}} =< Ty(\cdot), \varphi_i(\cdot) >_{W_1^{2m+1}} = (Ty)(x_i) = 0.$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense in $[0,b]$, hence $Ty(x) = 0$. It follows that $y(x) \equiv 0$ by the existence of T^{-1}. So the proof is complete. □

3 The Main Results

First we construct the iterative sequence $y_n(x)$. Putting an arbitrary initial value function $y_1(x) \in W_1^{2m+1}[0,b]$, let

$$\begin{cases} Tv_n(x) = g(x, y_{n-1}(x)),\\ y_n(x) = P_n v_n(x), \end{cases} \tag{3.1}$$

where $v_n(x) \in W_1^{2m+1}[0,b]$ is the solution of (3.1), $v_n^{(2l)}(0) = 0$, $v_n^{(2l)}(b) = 0$, $l = 0, 1, 2 \ldots, m-1$ and $P_n : W_1^{2m+1}[0,b] \to \mathrm{span}\{\overline{\psi}_1, \overline{\psi}_2, \ldots \overline{\psi}_n\}$ is the orthogonal projection operator. By (3.1), we obtain

$$\begin{cases} v_n(x) = \sum\limits_{k=0}^{2m-1} \alpha_k \frac{1}{k!} x^k + \underbrace{\int_0^x \cdots \int_0^x}_{2m} g(x, y_{n-1}(x))\, \underbrace{dx \ldots dx}_{2m},\\ v_n^{(1)}(x) = \sum\limits_{k=1}^{2m-2} \alpha_k \frac{1}{k\cdot k!} x^{k-1} + \underbrace{\int_0^x \cdots \int_0^x}_{2m-1} g(x, y_{n-1}(x))\, \underbrace{dx \ldots dx}_{2m-1},\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ v_n^{(2m)}(x) = g(x, y_{n-1}(x)),\\ v_n^{(2m+1)}(x) = \partial_x g(x, y_{n-1}(x)). \end{cases} \tag{3.2}$$

where $\alpha_{2l-1}, l = 1, 2 \ldots, m$ are constants that describe the boundary conditions at odd-order derivatives defined by

$$\alpha_1 = v^{(1)}(0), \alpha_3 = v^{(3)}(0), \ldots, \alpha_{2m-1} = v^{(2m-1)}(0),$$

and $\alpha_{2l}, l = 0, 1, 2, \ldots, m-1$ are the boundary conditions at even-order derivatives defined by

$$\alpha_0 = v(0),\ \alpha_2 = v^{(2)}(0),\ \alpha_4 = v^{(4)}(0), \ldots,\ \alpha_{2m-2} = v^{(2m-2)}(0),$$

which satisfy $\alpha_{2l} = 0, l = 0, 1, 2, \ldots, m-1$.

Lemma 2. *If* $\{x_i\}_{i=1}^{\infty}$ *is dense in* $[0, b]$*, then*

$$v_n(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} g(x_k, y_{n-1}(x_k)) \overline{\psi}_i(x).$$

Proof. Since functions $v_n(x) \in W_1^{2m+1}[0, b]$, $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ make the complete system in $W_1^{2m+1}[0, b]$, we have

$$\begin{aligned} v_n(x) &= \sum_{i=1}^{\infty} < v_n(x), \overline{\psi}_i(x) >_{W_1^{2m+1}} \overline{\psi}_i(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \overline{\psi}_i(x) \\ &\times < v_n(x), T^* \varphi_k(x) >_{W_1^{2m+1}} = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} < T v_n(x), \varphi_k(x) >_{W_1^{2m+1}} \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} g(x_k, y_{n-1}(x_k)) \overline{\psi}_i(x). \end{aligned}$$

The proof is complete. □

Taking an arbitrary initial value function $y_1(x) \in W_1^{2m+1}[0, b]$, let us define the iterative sequence

$$y_n(x) = P_n v_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} g(x_k, y_{n-1}(x_k)) \overline{\psi}_i(x), \tag{3.3}$$

where $P_n : W_1^{2m+1}[0, b] \to \text{span}\ \{\overline{\psi}_1, \overline{\psi}_2, \ldots \overline{\psi}_n\}$ is orthogonal projection operator.

3.1 The boundedness of sequence $y_n(x)$

Lemma 3. *Suppose that for* $x \in [0, b]$, $z \in (-\infty, +\infty)$, $g(x, z)$ *is a continuous bounded function, then* $\|v_n(x)\|_{W_1^{2m+1}}$ *is bounded.*

Proof. Note that $v_n^{(i)}(x) = \partial_x^i < v_n(\cdot),\ R_x(\cdot) >_{W_1^{2m+1}},\ i = 2m, 2m+1$, then we have

$$\|v_n\|^2_{W_1^{2m+1}} = \sum_{k=2m}^{2m+1} \int_0^b (v_n^{(k)}(x))^2 dx = \sum_{i=2m}^{2m+1} \int_0^b (\partial_x^i < v_n(\cdot), R_x(\cdot) >_{W_1^{2m+1}})^2 dx$$

$$= \sum_{i=2m}^{2m+1} \int_0^b \Big\{ \partial_x^i \Big[\int_0^b v_n^{(2m)}(\cdot)\partial_\cdot^{2m} R_x(\cdot) d\cdot + \int_0^b v_n^{(2m+1)}(\cdot)\partial_\cdot^{2m+1} R_x(\cdot) d\cdot \Big] \Big\}^2 dx$$

$$= \sum_{i=2m}^{2m+1} \int_0^b \Big\{ \partial_x^i \Big[\int_0^b v_n^{(2m)}(\cdot)\partial_\cdot^{2m} R_x(\cdot) d\cdot + v_n^{(2m)}(\cdot)\partial_\cdot^{2m+1}(\cdot)|_0^b$$

$$- \int_0^b v_n^{(2m)}(\cdot)\partial_\cdot^{2m+2} R_x(\cdot) d\cdot \Big] \Big\}^2 dx = \sum_{i=2m}^{2m+1} \int_0^b \Big\{ \partial_x^i \Big[\int_0^b v_n^{(2m)}(\cdot)\partial_\cdot^{2m} R_x(\cdot) d\cdot$$

$$+ \int_0^b \frac{1}{b}(v_n^{(2m)}(b)\partial_\cdot^{2m+1}(b) - v_n^{(2m)}(0)\partial_\cdot^{2m+1}(0)) d\cdot$$

$$- \int_0^b v_n^{(2m)}(\cdot)\partial_\cdot^{2m+2} R_x(\cdot) d\cdot \Big] \Big\}^2 dx = \sum_{i=2m}^{2m+1} \int_0^b \Big\{ \partial_x^i \Big[\int_0^b (v_n^{(2m)}(\cdot)\partial_\cdot^{2m} R_x(\cdot)$$

$$+ \frac{1}{b}(v_n^{(2m)}(b)\partial_\cdot^{2m+1}(b) - v_n^{(2m)}(0)\partial_\cdot^{2m+1}(0)) - v_n^{(2m)}(\cdot)\partial_\cdot^{2m+2} R_x(\cdot)) d\cdot \Big] \Big\}^2 dx$$

$$\le \sum_{i=2m}^{2m+1} b \int_0^b \int_0^b \Big[v_n^{(2m)}(\cdot)\partial_x^i\partial_\cdot^{2m} R_x(\cdot) - v_n^{(2m)}(\cdot)\partial_x^i\partial_\cdot^{2m+2} R_x(\cdot) \Big]^2 d\cdot\, dx$$

$$= \sum_{i=2m}^{2m+1} b \Big\{ \int_0^b \int_0^x \Big[v_n^{(2m)}(\cdot)\partial_x^i\partial_\cdot^{2m} R_x(\cdot) - v_n^{(2m)}(\cdot)\partial_x^i\partial_\cdot^{2m+2} R_x(\cdot) \Big]^2 d\cdot\, dx$$

$$+ \int_0^b \int_x^b \Big[v_n^{(2m)}(\cdot)\partial_x^i\partial_\cdot^{2m} R_x(\cdot) - v_n^{(2m)}(\cdot)\partial_x^i\partial_\cdot^{2m+2} R_x(\cdot) \Big]^2 d\cdot\, dx \Big\}.$$

In view of (2.1), we know $\partial_x^i\partial_\cdot^{2m} R_x(\cdot)$, $\partial_x^i\partial_\cdot^{2m+1} R_x(\cdot)$, $\partial_x^i\partial_\cdot^{2m+2} R_x(\cdot)$, $i = 2m, 2m+1$ are bounded as $x \neq \cdot$ in $[0, b]$. In terms of (3.2) and the assumptions, we know $v_n^{(2m)}(x)$ is bounded. Thus $\|v_n\|_{W_1^{2m+1}}$ is bounded. □

In the following sections, $C_k, k = 0, 1, 2, \ldots, 2m, M, M_1$ are constants.

Lemma 4. *Assume that for $x \in [0, b]$, $z \in (-\infty, +\infty)$, $g(x, z)$ is a continuous bounded function, then $\|y_n\|_{W_1^{2m+1}} \le M$.*

Proof. From Lemma 3, it follows that $\|v_n\|_{W_1^{2m+1}} \le M$. By (3.3) estimates $\|y_n\|_{W_1^{2m+1}} \le \|v_n\|_{W_1^{2m+1}} \le M$ hold. □

In the following discussions, we will prove that the solution $y(x)$ of Eq. (3.1) exists and $\|y_n^{(k)} - y^{(k)}\|_C \to 0$, $k = 0, 1, 2, \ldots 2m$ as $n \to \infty$.

3.2 The existence of the solution of (2.3) and convergence of $y_n(x)$

Several Lemmas are given first.

Lemma 5. *If $y(x) \in W_1^{2m+1}[0,b]$, then*

$$\|y^{(k)}\|_C \le C_k \|y\|_{W_1^{2m+1}}, \quad k = 0,1,2,\dots,2m.$$

Proof. For any $x, t \in [0,b]$, $y^{(k)}(x) =< y(t), \partial_x^k R_x(t) >_{W_1^{2m+1}}$. Note that

$$\left\|\partial_x^k R_x(t)\right\|_{W_1^{2m+1}} \le C_k, \quad k = 0,1,2,\dots,2m, \tag{3.4}$$

then

$$\begin{aligned}|y^{(k)}(x)| &= | < y(t), \partial_x^k R_x(t) >_{W_1^{2m+1}} | \le \|y(t)\|_{W_1^{2m+1}} \|\partial_x^k R_x(t)\|_{W_1^{2m+1}} \\ &\le C_k \|y\|_{W_1^{2m+1}},\end{aligned}$$

thus

$$\|y^{(k)}\|_C \le C_k \|y\|_{W_1^{2m+1}}, \quad k = 0,1,2,\dots,2m.$$

□

Lemma 6. *Suppose the conditions of Lemma 4 hold, then $\|y_n^{(k)}\|_C \le M_1, k = 0,1,2,\dots 2m$.*

Proof. From Lemma 4, $\|y_n\|_{W_1^{2m+1}} \le M$. By Lemma 5, we obtain $\|y_n^{(k)}\|_C \le C_k M$, $k = 0,1,2,\dots,2m$, thus $\|y_n^{(k)}\|_C \le M_1$, $k = 0,1,2,\dots,2m$. □

Lemma 7. *Suppose the conditions of Lemma 4 hold, then $\{y_n(x)\}_{n=1}^{\infty}$ is a compact set in space $C[0,b]$.*

Proof. By Lemma 4 it follows that $\|y_n\|_{W_1^{2m+1}} \le M$, from Lemma 6 we know that $\{y_n(x)\}_{n=1}^{\infty}$ is a bounded set in space $C[0,b]$. For an arbitrary $y_n(x)$,

$$\begin{aligned}|y_n(x+t) - y_n(x)| &= | < y_n(s), R_{x+t}(s) - R_x(s) >_{W_1^{2m+1}} | \le \|y_n\|_{W_1^{2m+1}} \\ &\times \|R_{x+t}(s) - R_x(s)\|_{W_1^{2m+1}} \le M \|\partial_x R_x(s) \mid_{x=\xi\in[x,x+t]} \|_{W_1^{2m+1}} t \le M C_1 t.\end{aligned}$$

Therefore, for any $\varepsilon > 0$, taking $\delta = \varepsilon / MC_1 > 0$ and $| \; t \; | \le \delta$, we obtain $|y_n(x+t) - y_n(x)| < \varepsilon$, so $y_n(x)$ is equicontinuous function with respect to n. Combining the above argument, $\{y_n(x)\}_{n=1}^{\infty}$ is a compact set in space $C[0,b]$. □

Theorem 1. *If $\{x_i\}_{i=1}^{\infty}$ is dense in $[0,b]$, for $x \in [0,b], z \in (-\infty, +\infty)$, $g(x,z)$ is continuous bounded function, $z = z(x), g(x,z) \in W_1^1[0,b]$, then there exists a subsequence $\{y_{n_p}(x)\}_{p=1}^{\infty}$ of $\{y_n(x)\}_{n=1}^{\infty}$ and $\overline{y}(x) \in C^2[0,b]$ such that*

$$\|y_{n_p}^{(k)} - \overline{y}^{(k)}\|_C \to 0 \;, k = 0,1,2,\dots,2m \quad as \quad p \to \infty \tag{3.5}$$

where $\overline{y}(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} g(x_k, \overline{y}(x_k)) \overline{\psi}_i(x)$.

Proof. By Lemma 4, $\|y_n\|_{W_1^{2m+1}} \le M$, we infer that $\{y_n(x)\}_{n=1}^{\infty}$ is a compact set in space $C[0,b]$ from Lemma 6, hence there exist $\overline{y}(x) \in C^2[0,b]$ and a convergent subsequence $\{y_{n_p}(x)\}_{p=1}^{\infty}$ of $\{y_n(x)\}_{n=1}^{\infty}$ such that

$$\overline{y}(x) = \lim_{p\to\infty} y_{n_p}(x) = \lim_{p\to\infty} \sum_{i=1}^{n_p} \sum_{k=1}^{i} \beta_{ik} g(x_k, y_{n_p-1}(x_k))\overline{\psi}_i(x).$$

One gets $\overline{y}(x) = \sum\limits_{i=1}^{\infty} \sum\limits_{k=1}^{i} \beta_{ik} g(x_k, \overline{y}(x_k))\overline{\psi}_i(x)$ with respect to x, uniformly. And by Lemma 6, $\|y_{n_p}^{(k)}\|_C \le M_1, k = 0,1,2,\ldots,2m$, then there exists a subsequence $\{y_{n_{p_j}}(x)\}_{j=1}^{\infty}$ of $\{y_{n_p}(x)\}_{n=1}^{\infty}$ such that

$$\|y_{n_{p_j}}^{(k)} - \overline{y}^{(k)}\|_C \to 0\ ,\ k = 0,1,2,\ldots,2m \quad \text{as} \ \ j \to \infty.$$

Without loss of generality, we write $y_{n_{p_j}}(x)$ with $y_{n_p}(x)$, consequently,

$$\|y_{n_p}^{(k)} - \overline{y}^{(k)}\|_C \to 0\ ,\ k = 0,1,2,\ldots,2m \quad \text{as} \ \ p \to \infty$$

hold. □

Corollary 1. If $\|y_n\|_{W_1^{2m+1}} \le M$, then there exists a subsequence $\{y_{n_p}(x)\}_{p=1}^{\infty}$ of $\{y_n(x)\}_{n=1}^{\infty}$ and $\overline{y}(x) \in C^2[0,b]$ such that

$$\|y_{n_p}^{(k)} - \overline{y}^{(k)}\|_C \to 0\ ,\ k = 0,1,2,\ldots,2m \quad \text{as} \ \ p \to \infty$$

where $\overline{y}(x) = \sum\limits_{i=1}^{\infty} \sum\limits_{k=1}^{i} \beta_{ik} g(x_k, \overline{y}(x_k))\overline{\psi}_i(x)$.

Next we will prove that $\overline{y}(x) \in W_1^{2m+1}[0,b]$, $\overline{y}(x)$ is the solution of Eq.(2.3).

Lemma 8. $\overline{y}(x)$ *is absolutely continuous function.*

Proof. For $\overline{y}(x)$ and arbitrary $\alpha_i, \beta_i \in [0,b]$, when $\sum\limits_{i=1}^{n} |\alpha_i - \beta_i| < \delta = \varepsilon/MC_1$, we have

$$\sum_{i=1}^{n} \big|\overline{y}(\alpha_i) - \overline{y}(\beta_i)\big| = \sum_{i=1}^{n} \big| \lim_{p\to\infty} y_{n_p}(\alpha_i) - \lim_{p\to\infty} y_{n_p}(\beta_i)\big| \sum_{i=1}^{n} \big| \lim_{p\to\infty} (y_{n_p}(\alpha_i)$$
$$- y_{n_p}(\beta_i))\big| = \sum_{i=1}^{n} \big| \lim_{p\to\infty} < y_{n_p}(\eta), R_{\alpha_i}(\eta) - R_{\beta_i}(\eta) >_{W_1^{2m+1}} \big|$$
$$= \sum_{i=1}^{n} \lim_{p\to\infty} \big| < y_{n_p}(\eta), R_{\alpha_i}(\eta) - R_{\beta_i}(\eta) >_{W_1^{2m+1}} \big|$$

$$\leq \sum_{i=1}^{n} \lim_{p\to\infty} \left\|y_{n_p}(\eta)\right\|_{W_1^{2m+1}} \left\|R_{\alpha_i}(\eta) - R_{\beta_i}(\eta)\right\|_{W_1^{2m+1}}$$

$$\leq M\sum_{i=1}^{n} \left\|R_{\alpha_i}(\eta)-R_{\beta_i}(\eta)\right\|_{W_1^{2m+1}} = M\sum_{i=1}^{n} \left\|\partial_x R_x(\eta)\big|_{x=\zeta\in[\alpha_i,\beta_i]}(\alpha_i-\beta_i)\right\|_{W_1^{2m+1}}$$

$$< MC_1 \sum_{i=1}^{n} \left|\alpha_i - \beta_i\right| < MC_1\delta = \varepsilon,$$

where C_1 are given by (3.4). So $\overline{y}(x)$ is absolutely continuous function. □

Theorem 2. *Suppose that the conditions of Theorem 1 hold, $\overline{y}(x)$ is given by* (3.5)*, then $\overline{y}(x) \in W_1^{2m+1}[0,b]$, $\overline{y}(x)$ is the solution of Eq.* (2.3)*.*

Proof. By Lemma 8, $g(x,\overline{y}(x))$ is absolutely continuous, furthermore, the derivative $\partial_x g(x,\overline{y}(x)) \in L^2[0,b]$. In view of Definition 3, $g(x,\overline{y}(x)) \in W_2^1[0,b]$. In consequence $T^{-1}g(x,\overline{y}(x)) \in W_1^{2m+1}[0,b]$,

$$\begin{aligned}
T^{-1}g(x,\overline{y}(x)) &= \sum_{i=1}^{\infty} < T^{-1}g(x,\overline{y}(x)), \overline{\psi}_i(x) >_{W_1^{2m+1}} \overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} < T^{-1}g(x,\overline{y}(x)), \psi_k(x) >_{W_1^{2m+1}} \overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} < T^{-1}g(x,\overline{y}(x)), T^*\varphi_k(x) >_{W_1^{2m+1}} \overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} < TT^{-1}g(x,\overline{y}(x)), \varphi_k(x) >_{W_2^{2m+1}} \overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} g(x_k,\overline{y}(x_k))\overline{\psi}_i(x).
\end{aligned}$$

In terms of (3.5), we get $\overline{y}(x) = T^{-1}g(x,\overline{y}(x))$, i.e., $\overline{y}(x) \in W_1^{2m+1}[0,b]$. Then the equality $T\overline{y}(x) = g(x,\overline{y}(x))$ holds. Hence, $\overline{y}(x)$ is the solution of Eq.(2.3). □

Remark 1. Under the conditions of Theorem 1, the solution of (2.3) exists and it satisfies $y(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} g(x_k, y(x_k))\overline{\psi}_i(x)$.

Corollary 2. Assume that the conditions of Theorem 1 hold. If $\{y_{n_p}(x)\}_{p=1}^{\infty}$ is an arbitrary convergent subsequence of $\{y_n(x)\}_{n=1}^{\infty}$, then the limit function of $\{y_{n_p}(x)\}_{p=1}^{\infty}$ must be the solution of (2.3).

Theorem 3. *Suppose the solution $y(x)$ of Eq.* (2.3) *exists and is unique, the conditions of Theorem 1 hold, then*

$$\|y_n^{(k)} - y^{(k)}\|_C \to 0 \ , k \ = 0,1,2,\ldots,2m \quad as \quad n \to \infty,$$

where $y_n(x)$ is given by (3.3)*.*

Proof. Assume that $\{y_n(x)\}_{n=1}^{\infty}$ doesn't converge to $y(x)$, then there exists a $\varepsilon_0 > 0$ and a subsequence $\{y_{n_p}(x)\}_{p=1}^{\infty}$ of $\{y_n(x)\}_{n=1}^{\infty}$ such that $\|y_{n_p} - y\|_C \geq \varepsilon_0$. On the other hand, by Lemma 4, $\| y_n \|_{W_1^{2m+1}} \leq M$, further $\| y_{n_p} \|_{W_1^{2m+1}} \leq M$, thus there exists a subsequence $\{y_{n_{p_j}}(x)\}_{j=1}^{\infty}$ of $\{y_{n_p}(x)\}_{p=1}^{\infty}$ such that

$$y_{n_{p_j}}(x) \to y^*(x) \quad j \to \infty$$

uniformly. From Corollary 2, $y^*(x)$ is also the solution of (2.3). But $y(x) \neq y^*(x)$, this is a contradictory conclusion with the uniqueness of the solution of Eq. (2.3). Consequently, $\|y_n - y\|_C \to 0$ as $n \to \infty$. In the same way, we can verify

$$\|y_n^{(k)} - y^{(k)}\|_C \to 0\ ,\ k = 1, 2, \ldots, 2m \ \text{ as } \ n \to \infty.$$

□

Corollary 3. If $\| y_n \|_{W_1^{2m+1}} \leq M$, then $\|y_n^{(k)} - y^{(k)}\|_C \to 0$, $k = 0, 1, 2, \ldots, 2m$ as $n \to \infty$, where $y_n(x)$ is given by (3.3).

Remark 2. In computational experiments, it is easy to test $\| y_n \|_{W_1^{2m+1}} \leq M$.

4 Numerical Results

All computations were carried out using Mathematica 5.0.

Example 1. Consider the following 6th-order boundary value problem (in [14], Example 2)

$$\begin{cases} y^{(6)}(x) = e^{-x} y^2(x), \quad x \in (0,1), \\ y(0) = y''(0) = y^{(4)}(0) = 1, \\ y(1) = y''(1) = y^{(4)}(1) = e, \end{cases}$$

with the exact solution $y(x) = e^x$. We use $n = 15$. In Table 1, our results are compared with the results in [14]. They show that our method is superior to the one in [14].

Table 1. Absolute errors for example 1. AE denote absolute error in our method; AE [14] denote absolute error in [14].

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
AE 10^{-7}	1.070	2.004	2.705	3.115	3.212	3.003	2.519	1.811	9.463E-1	0.
AE[14] 10^{-4}	1.233	2.354	3.257	3.855	4.086	3.919	3.36	2.459	1.299	2.000E-5

From the presented results it follows that our method is superior to one in [14].

In Table 2, we give the root-mean-square errors of $y^{(i)}, i = 0, 1, 2, \ldots, 6$.

Table 2. Root-mean-square errors for example 1.Note: $\mathrm{RMS}_{(i)}$ denote the root-mean square error of $y^{(i)}, i = 0, 1, 2, \ldots, 6$.

$\mathrm{RMS}_{(0)}$	$\mathrm{RMS}_{(1)}$	$\mathrm{RMS}_{(2)}$	$\mathrm{RMS}_{(3)}$	$\mathrm{RMS}_{(4)}$	$\mathrm{RMS}_{(5)}$	$\mathrm{RMS}_{(6)}$
2.35E-7	6.92E-7	2.34E-6	6.87E-6	2.81E-5	4.87E-5	1.07E-3

Example 2. Consider the following 8th-order boundary value problem (in [19], Example 4)

$$\begin{cases} y^{(8)}(x) = e^{-x}y^2(x), \quad x \in (0,1), \\ y(0) = y''(0) = y^{(4)}(0) = y^{(6)}(0) = 1, \\ y(1) = y''(1) = y^{(4)}(1) = y^{(6)}(1) = e, \end{cases}$$

with exact solution $y(x) = e^x$. We use $n = 15$. In Table 3, our results are compared with the results in [19].

Table 3. Absolute errors for Example 2.

x	our method Error	method in [19] Error1	method in [19] Error2
0.25	2.33E-8	1.00E-4	4.91E-5
0.5	3.25E-8	1.43E-4	7.04E-5
0.75	2.28E-8	9.91E-5	4.98E-5

Table 4. Root-mean-square errors for example 2. Note:$\mathrm{RMS}_{(i)}$ denote the root-mean square error of $y^{(i)}, i = 0, 1, 2, \ldots, 8$.

$\mathrm{RMS}_{(0)}$	$\mathrm{RMS}_{(1)}$	$\mathrm{RMS}_{(2)}$	$\mathrm{RMS}_{(3)}$	$\mathrm{RMS}_{(4)}$	$\mathrm{RMS}_{(5)}$	$\mathrm{RMS}_{(6)}$	$\mathrm{RMS}_{(7)}$	$\mathrm{RMS}_{(8)}$
2.38E-8	6.98E-8	2.34E-7	6.89E-7	2.33E-6	6.84E-6	2.79E-5	4.87E-5	1.07E-3

In Table 4, we give the root-mean-square errors of $y^{(i)}, i = 0, 1, 2, \ldots, 8$.

Example 3. Consider the following 8th-order boundary value problem (in [19], Example 3)

$$\begin{cases} y^{(8)}(x) = e^{-x}y^2(x), \quad x \in (0,1), \\ y^{(i)}(0) = 1, \ i = 0, 1, 2, \ldots, 7, \end{cases}$$

with exact solution $y(x) = e^x$. We use $n = 50$. In Table 5, our results are compared with the results in [19].

In Table 6, we give the root-mean-square errors of $y^{(i)}, i = 0, 1, 2, \ldots, 8$.

Example 4. Consider the following 10th-order boundary value problem

$$\begin{cases} y^{(10)}(x) = (x+1)e^{-y(x)}, \quad x \in (0,1), \\ y(0) = y''(0) = y^{(4)}(0) = y^{(6)}(0) = y^{(8)}(0) = 0, \\ y(1) = -y''(1) = y^{(4)}(1) = -y^{(6)}(1) = y^{(8)}(1) = \sin(1), \end{cases}$$

Table 5. Absolute errors for Example 3.

x	our method Error	method in [19] Error1	method in [19] Error2
0.25	2.0E-12	2.1E-11	1.0E-11
0.5	2.8E-10	1.0E-8	5.1E-9
0.75	4.9E-9	4.0E-7	1.9E-7
1.0	3.7E-8	5.5E-6	2.5E-6

Table 6. Root-mean-square errors for example 3. Note:$\text{RMS}_{(i)}$ denote the root-mean square error of $y^{(i)}, i = 0, 1, 2, \ldots, 8$.

$\text{RMS}_{(0)}$	$\text{RMS}_{(1)}$	$\text{RMS}_{(2)}$	$\text{RMS}_{(3)}$	$\text{RMS}_{(4)}$	$\text{RMS}_{(5)}$	$\text{RMS}_{(6)}$	$\text{RMS}_{(7)}$	$\text{RMS}_{(8)}$
9.12E-9	6.97E-8	4.62E-7	2.60E-6	1.20E-5	4.33E-5	1.13E-4	1.99E-4	5.54E-5

with exact solution $y(x) = \sin x$. We use $n = 15$. In Table 7, we give the root-mean-square errors of $y^{(i)}, i = 0, 1, 2, \ldots, 10$.

Table 7. Root-mean-square errors for example 4. $\text{RMS}_{(i)}$ denote the root-mean square error of $y^{(i)}, i = 0, 1, 2, \ldots, 10$.

$\text{RMS}_{(0)}$	$\text{RMS}_{(1)}$	$\text{RMS}_{(2)}$	$\text{RMS}_{(3)}$	$\text{RMS}_{(4)}$	$\text{RMS}_{(5)}$	$\text{RMS}_{(6)}$	$\text{RMS}_{(7)}$	$\text{RMS}_{(8)}$	$\text{RMS}_{(9)}$	$\text{RMS}_{(10)}$
1.03E-9	3.04E-9	1.02E-8	3.0E-8	1.01E-7	3.03E-7	1.07E-6	3.8E-6	2.11E-5	1.07E-4	1.14E-3

5 Discussion and Conclusion

In this paper, we established the existence of the solution and a new iterative algorithm for the high-order boundary value problems in reproducing kernel space. The iterative method is convergent for arbitrary initial value function $y_1(x)$, therefore, it is a large-range convergence iterative method. The approximate solution $y_n(x)$ and the exact solution $y(x)$ satisfy $\|y_n - y\|_C \to 0$ as $n \to \infty$. Furthermore, we verify that $\|y_n^{(k)} - y^{(k)}\|_C \to 0, k = 1, 2, \ldots, 2m$ as $n \to \infty$, the numerical results illustrate the accuracy of our method.

6 Appendix

By (2.1), we have

$$\begin{aligned} < y(t), R_x(t) >_{W_1^{2m+1}} = & \int_0^b y(t)(R_x^{(4m)}(t) - R_x^{(4m+2)}(t))\, dt \\ & + \Big[\sum_{i=1}^{2m-1} (-1)^i y^{(i-1)}(t)(R_x^{(4m-i)}(t) - R_x^{(4m+2-i)}(t)) \Big] \Big|_0^b \\ & + \Big[y^{(2m)}(t) R_x^{(2m+1)}(t) \Big] \Big|_0^b . \end{aligned}$$

Since $R_x(t) \in W_1^{2m+1}[0,b]$, it follows that

$$R_x^{(2l)}(0) = 0, \quad R_x^{(2l)}(b) = 0, \quad l = 0, 1, 2, \dots, m-1. \tag{6.1}$$

If $R_x(t)$ satisfies

$$R_x^{(4m)}(t) - R_x^{(4m+2)}(t) = \delta(t-x) \tag{6.2}$$

and the following differential equations:

$$\begin{cases} R_x^{(4m-j)}(0) - R_x^{(4m+2-j)}(0) = 0, \quad j = 2, 4, \dots, 2m, \\ R_x^{(4m-j)}(b) + R_x^{(4m+2-j)}(b) = 0, \quad j = 2, 4, \dots, 2m, \\ R_x^{(2m+1)}(0) = 0, \quad R_x^{(2m+1)}(b) = 0, \end{cases} \tag{6.3}$$

then $(y(t), R_x(t))_{W_1^{2m+1}} = y(x)$. Obviously, $R_x(t)$ is the reproducing kernel of $W_1^{2m+1}[0,b]$.

In the following, we will get the expression of the reproducing kernel $R_x(t)$. Note that characteristic equation of (6.2) is given by $\lambda^{4m}(\lambda^2 - 1) = 0$, and characteristic values are $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 0$, where λ_3 is a multiple root. We present the reproducing kernel $R_x(t)$ by

$$R_x(t) = \begin{cases} \sum_{i=1}^{4m} a_i t^{i-1} + a_{4m+1} e^t + a_{4m+2} e^{-t}, \quad t \le x, \\ \sum_{i=1}^{4m} b_i t^{i-1} + b_{4m+1} e^t + b_{4m+2} e^{-t}, \quad t > x. \end{cases} \tag{6.4}$$

On the other hand, for $R_x(t) \in W_1^{2m+1}[0,b]$, let $R_x(t)$ satisfy

$$R_x^{(k)}(x+0) = R_x^{(k)}(x-0), \quad k = 0, 1, 2, \dots, 4m. \tag{6.5}$$

Integrating (6.2) from $x - \varepsilon$ to $x + \varepsilon$ with respect to t and let $\varepsilon \to 0$ (we have the jump degree of $R_x^{(4m+1)}(t)$ at $t = x$) one obtains

$$R_x^{(4m+1)}(x-0) - R_x^{(4m+1)}(x+0) = 1. \tag{6.6}$$

Through (6.1), (6.3), (6.5), (6.6), the unknown coefficients of (2.1) can be obtained. Similarly, we can obtain the unknown coefficients of (2.2).

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