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Classical Solution of the Cauchy Problem for Biwave Equation: Application of Fourier Transform

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Abstract. In this paper, we use some Fourier analysis techniques to find an exact solution to the Cauchy problem for the n-dimensional biwave equation in the upper half-space $\mathbb{R}^n \times [0, +\infty)$.

Keywords: biwave equation, Fourier transform, Cauchy problem.

AMS Subject Classification: 35G05; 35G10.

1 Introduction

The Cauchy initial value problem for the n-dimensional biwave equation consists in finding a scalar function $u \in C^4(\mathbb{R}^n \times [0, +\infty))$ such that for $(x, t) \in \mathbb{R}^n \times (0, +\infty)$ then

$$\left(\frac{\partial^2}{\partial t^2} - a^2\Delta\right)\left(\frac{\partial^2}{\partial t^2} - b^2\Delta\right) u(x, t) = f(x, t), \quad a^2 > b^2 > 0, \tag{1.1}$$

together with the initial conditions

$$u(x, 0) = \phi_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = \phi_1(x), \quad \frac{\partial^2 u}{\partial t^2}(x, 0) = \phi_2(x), \quad \frac{\partial^3 u}{\partial t^3}(x, 0) = \phi_3(x), \tag{1.2}$$

for $(x, t) \in \mathbb{R}^n \times \{0\}$.

The biwave equation has been studied in some models related to the mathematical theory of elasticity. Let us consider the mathematical formulation for the displacement equation of a homogeneous isotropic elastic body. Remark that, the Newton's second law leads to the Cauchy's motion equation of an elastic body, which takes the form

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \mathbf{f} = \rho \ddot{\mathbf{u}}, \tag{1.3}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor field, $\mathbf{u}$ is the displacement vector field, $\mathbf{f}$ is the vector field of body force per unit volume and ρ is the mass density.

The infinitesimal strain tensor field is given by the equation

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^T\right]. \tag{1.4}$$

Moreover, the Hooke's law for homogeneous isotropic bodies has the form

$$\boldsymbol{\sigma} = \lambda\,\mathrm{trace}(\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}, \tag{1.5}$$

where $\lambda, \mu > 0$ are Lamé's parameters and $\mathbf{I}$ is the second-order identity tensor. Substituting the strain-displacement equation (1.4) and the Hooke's equation (1.5) into the equilibrium equation (1.3), we obtain the Navier's elastodynamic wave equation

$$(\lambda + \mu)\nabla\,\mathrm{div}(\mathbf{u}) + \mu\Delta\mathbf{u} + \mathbf{f} = \rho\ddot{\mathbf{u}}. \tag{1.6}$$

This equation in the Cartesian coordinates has the form

$$(\lambda + \mu)\frac{\partial}{\partial x_k}\left(\sum_{j=1}^{n}\frac{\partial u_j}{\partial x_j}\right) + \mu\Delta u_k + f_k = \rho\frac{\partial^2 u_k}{\partial t^2}, \quad k = 1, \dots, n.$$

Let us denote $a^2 = (\lambda + 2\mu)/\rho$, $b^2 = \mu/\rho$, then (1.6) can be rewritten as

$$\mathcal{L} \equiv \left(\frac{\partial^2}{\partial t^2} - b^2\Delta\right)\mathbf{u} - (a^2 - b^2)\nabla\,\mathrm{div}(\mathbf{u}) - \frac{\mathbf{f}}{\rho} = 0. \tag{1.7}$$

It is easy to show that the equation (1.7) has a solution in the following form

$$\mathbf{u} = \left(\frac{\partial^2}{\partial t^2} - a^2\Delta\right)\mathbf{w} + (a^2 - b^2)\nabla\,\mathrm{div}(\mathbf{w}), \tag{1.8}$$

where $\mathbf{w}$ is a solution to the biwave equation

$$\left(\frac{\partial^2}{\partial t^2} - a^2\Delta\right)\left(\frac{\partial^2}{\partial t^2} - b^2\Delta\right)\mathbf{w} = \frac{\mathbf{f}}{\rho}. \tag{1.9}$$

This formula is called as Cauchy–Kovalevski–Somigliana solution to the elastodynamic wave equation. Indeed, substituting (1.8)–(1.9) to the left-hand side of (1.7), we have

$$\mathcal{L} = \left(\frac{\partial^2}{\partial t^2} - b^2\Delta\right)\left(\left(\frac{\partial^2}{\partial t^2} - a^2\Delta\right)\mathbf{w} + (a^2 - b^2)\nabla\,\mathrm{div}(\mathbf{w})\right)$$
$$- (a^2 - b^2)\nabla\left(\left(\frac{\partial^2}{\partial t^2} - a^2\Delta\right)\mathrm{div}(\mathbf{w}) + (a^2 - b^2)\Delta\,\mathrm{div}(\mathbf{w})\right) - \frac{\mathbf{f}}{\rho}.$$

Note that

$$\left(\frac{\partial^2}{\partial t^2} - a^2\Delta\right)\operatorname{div}(\mathbf{w}) + (a^2 - b^2)\Delta\operatorname{div}(\mathbf{w}) = \left(\frac{\partial^2}{\partial t^2} - b^2\Delta\right)\operatorname{div}(\mathbf{w}).$$

Therefore, we get

$$\mathcal{L} = \left(\frac{\partial^2}{\partial t^2} - b^2\Delta\right)\left(\frac{\partial^2}{\partial t^2} - a^2\Delta\right)\mathbf{w} + (a^2 - b^2)\left(\frac{\partial^2}{\partial t^2} - b^2\Delta\right)\nabla\operatorname{div}(\mathbf{w})$$
$$- (a^2 - b^2)\nabla\left(\frac{\partial^2}{\partial t^2} - b^2\Delta\right)\operatorname{div}(\mathbf{w}) - \frac{\mathbf{f}}{\rho} = 0\,.$$

For more explanations about physical context, we refer the reader to [5, 8, 10]. Actually, there are not many mathematical papers related to biwave equations because it gets more difficult when studying high-order PDEs. In some recent researches, the symmetry analysis of biwave equations is considered and exact solutions are obtained by Fushchych, Roman and Zhdanov [3]; the existence and uniqueness of the solution to Cauchy initial value problem, bounded valued problem are given by Korzyuk, Cheb and Konopelko [6, 7]; the finite element methods for approximations of biwave equation are developed by Feng and Neilan [1, 2]. In our present work, the main result is to show the exact classical solution to the Cauchy initial value problem for the n-dimensional biwave equation by using some techniques of Fourier analysis.

Returning to the Cauchy problem for the biwave equation (1.1), we suppose that ϕ_0, ϕ_1, ϕ_2, ϕ_3, and f are elements in Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions on $\mathbb{R}^n$. Remark that, an indefinitely differentiable function ϕ is called rapidly decreasing when ϕ and all its derivatives are required to satisfy that

$$\|\phi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n}\left|x^\alpha\left(\frac{\partial}{\partial x}\right)^\beta\phi(x)\right| < \infty,$$

for every multi-index α and β. The Fourier transform of Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\mathcal{F}\left[\phi\right](\xi) \equiv \widehat{\phi}\,(\xi) = \int_{\mathbb{R}^n} e^{-ix.\xi}\phi(x)\,dx.$$

The convolution of two integrable functions ϕ and ψ is written as $\phi * \psi$. It is defined as the integral of the product of the two functions after one is reversed and shifted. As such, it is a particular kind of integral transform:

$$\left(\phi * \psi\right)(t) = \int_{\mathbb{R}^n}\phi\,(\tau)\psi(t-\tau)\,d\tau.$$

In the Euclidean space $\mathbb{R}^n$, the spherical mean of an integrable function ϕ around a point x is the average of all values of that function on a sphere of radius R centered at that point, i.e. it is defined by the formula

$$\mathcal{M}_R(\phi)(x) = \frac{1}{\omega_n R^{n-1}}\int\limits_{\partial B(x,R)}\phi(y)\,d\sigma(y) \equiv \frac{1}{\omega_n}\int\limits_{\partial B(0,1)}\phi(x+Ry)\,d\sigma(y),$$

where ω_n is the surface area of the n-dimensional unit ball and σ is the spherical measure area.

2 Main Results

The Cauchy problem for the homogeneous biwave equation in $\mathbb{R}^n \times [0,+\infty)$ that we will be studying in this section, reads as follows

$$\left(\frac{\partial^2}{\partial t^2} - a^2\Delta\right)\left(\frac{\partial^2 u}{\partial t^2} - b^2\Delta u\right) = 0, \quad a^2 > b^2 > 0, \tag{2.1}$$

with the initial conditions

$$u|_{t=0} = \phi_0(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \phi_1(x), \quad \left.\frac{\partial^2 u}{\partial t^2}\right|_{t=0} = \phi_2(x), \quad \left.\frac{\partial^3 u}{\partial t^3}\right|_{t=0} = \phi_3(x), \tag{2.2}$$

where ϕ_0, ϕ_1, ϕ_2, ϕ_3 are Schwartz functions.

The equation (2.1) can be rewritten as a fourth-order PDE, which has the following form

$$\frac{\partial^4 u}{\partial t^4} - (a^2+b^2)\frac{\partial^2}{\partial t^2}\Delta u + a^2b^2\Delta^2 u = 0. \tag{2.3}$$

Taking Fourier transform to the both sides of the equation (2.3), we obtain

$$\frac{\partial^4}{\partial t^4}\widehat{u}(\xi,t) + (a^2+b^2)|\xi|^2\frac{\partial^2}{\partial t^2}\widehat{u}(\xi,t) + a^2b^2|\xi|^4\widehat{u}(\xi,t) = 0.$$

This fourth ODE has the general solution, which takes the form

$$\widehat{u}(\xi,t) = C_1\cos(a|\xi|t) + C_2\sin(a|\xi|t) + C_3\cos(b|\xi|t) + C_4\sin(b|\xi|t),$$

where parameters C_1, C_2, C_3, C_4 are determined from the initial conditions:

$$\widehat{u}(\xi,t)|_{t=0} = C_1 + C_3 = \widehat{\phi_0}(\xi), \qquad \left.\frac{\partial\widehat{u}(\xi,t)}{\partial t}\right|_{t=0} = aC_2|\xi| + bC_4|\xi| = \widehat{\phi_1}(\xi),$$

$$\left.\frac{\partial^2\widehat{u}(\xi,0)}{\partial t^2}\right|_{t=0} = -a^2C_1|\xi|^2 - b^2C_3|\xi|^2 = \widehat{\phi_2}(\xi),$$

$$\left.\frac{\partial^3\widehat{u}(\xi,0)}{\partial t^3}\right|_{t=0} = -a^3C_2|\xi|^3 - b^3C_4|\xi|^3 = \widehat{\phi_3}(\xi).$$

Solving above system of equations, we easily get the image of solution $u(x,t)$ via Fourier transform given by

$$\widehat{u}(\xi,t) = -\frac{b^2|\xi|^2\widehat{\phi_0}(\xi) + \widehat{\phi_2}(\xi)}{(a^2-b^2)|\xi|^2}\cos(a|\xi|t) - \frac{b^2|\xi|^2\widehat{\phi_1}(\xi) + \widehat{\phi_3}(\xi)}{(a^3-ab^2)|\xi|^3}\sin(a|\xi|t)$$
$$+ \frac{a^2|\xi|^2\widehat{\phi_0}(\xi) + \widehat{\phi_2}(\xi)}{(a^2-b^2)|\xi|^2}\cos(b|\xi|t) + \frac{a^2|\xi|^2\widehat{\phi_1}(\xi) + \widehat{\phi_3}(\xi)}{(a^2b-b^3)|\xi|^3}\sin(b|\xi|t),$$

or by the rewritten form

$$\begin{aligned}\widehat{u}(\xi,t) = &-\frac{b^2}{a^2-b^2}\widehat{\phi_0}(\xi)\cos(a|\xi|t) + \frac{a^2}{a^2-b^2}\widehat{\phi_0}(\xi)\cos(b|\xi|t)\\ &-\frac{b^2}{a(a^2-b^2)}\widehat{\phi_1}(\xi)\frac{\sin(a|\xi|t)}{|\xi|} + \frac{a^2}{b(a^2-b^2)}\widehat{\phi_1}(\xi)\frac{\sin(b|\xi|t)}{|\xi|}\\ &+\frac{\widehat{\phi_3}(\xi)}{a^2-b^2}\left[\frac{1}{b}\frac{\sin(b|\xi|t)}{|\xi|^3} - \frac{1}{a}\frac{\sin(a|\xi|t)}{|\xi|^3}\right]\\ &+\frac{\widehat{\phi_2}(\xi)}{a^2-b^2}\left[\frac{\cos(b|\xi|t)}{|\xi|^2} - \frac{\cos(a|\xi|t)}{|\xi|^2}\right]. \end{aligned} \tag{2.4}$$

In the next sequence, we will find the inverse formula of (2.4) and obtain an exact solution to the equation (2.1).

Theorem 1. *The Cauchy problem for the homogeneous biwave equation in* $\mathbb{R}\times[0,+\infty)$ *has the following solution*

$$\begin{aligned}u(x,t) = \frac{1}{2ab(a^2-b^2)}\Bigg[&-b^3\int_{x-at}^{x+at}\phi_1(y)\,dy + a^3\int_{x-bt}^{x+bt}\phi_1(y)\,dy\\ &-ab\int_{x-at}^{x-bt}\int_0^y \phi_2(u)\,du\,dy + ab\int_{x+bt}^{x+at}\int_0^y\phi_2(u)\,du\,dy\\ &+b\int_{x-at}^{x+at}\int_0^y\int_0^\tau \phi_3(\omega)\,d\omega\,d\tau\,dy - a\int_{x-bt}^{x+bt}\int_0^y\int_0^\tau\phi_3(\omega)\,d\omega\,d\tau\,dy\\ &-ab^3\phi_0(x+at) - ab^3\phi_0(x-at) + a^3b\phi_0(x+bt) + a^3b\phi_0(x-bt)\Bigg].\end{aligned} \tag{2.5}$$

Proof. We have that

$$\begin{aligned}\cos(a|\xi|t) &= \frac{e^{ia|\xi|t}+e^{-ia|\xi|t}}{2}, \qquad \sin(a|\xi|t) = \frac{e^{ia|\xi|t}-e^{-ia|\xi|t}}{2i},\\ \frac{\sin(a|\xi|t)}{|\xi|} &= \frac{e^{ia|\xi|t}-e^{-ia|\xi|t}}{2i|\xi|} = \frac{1}{2}\int_{-at}^{at} e^{i|\xi|\theta}\,d\theta,\\ \frac{\cos(a|\xi|t)}{|\xi|^2} &= \frac{e^{ia|\xi|t}+e^{-ia|\xi|t}}{2|\xi|^2}\\ &= -\frac{1}{2}\int_0^{at}\int_0^y e^{i|\xi|u}\,du\,dy - \frac{1}{2}\int_0^{at}\int_0^y e^{-i|\xi|u}\,du\,dy + \frac{1}{|\xi|^2},\\ \frac{\sin(a|\xi|t)}{|\xi|^3} &= \frac{e^{ia|\xi|t}-e^{-ia|\xi|t}}{2i|\xi|^3} = -\frac{1}{2}\int_{-at}^{at}\int_0^y\int_0^\tau e^{i|\xi|u}\,du\,d\tau\,dy + \frac{at}{|\xi|^2}.\end{aligned}$$

Moreover,

$$\widehat{\delta}(x-\alpha t) = \int_{-\infty}^{+\infty} e^{-i|\xi|x}\delta(x-\alpha t)\,dx = e^{-i\alpha|\xi|t},$$

where $\delta(x)$ is the Dirac delta function. Hence, by the property of Dirac's delta function, we note that

$$\left(\frac{e^{ia|\xi|t}+e^{-ia|\xi|t}}{2}\right)\widehat{\phi_0}(\xi)=\frac{\mathcal{F}\left[\left(\delta(x+at)*\phi_0(x)\right)\right]+\mathcal{F}\left[\left(\delta(x-at)*\phi_0(x)\right)\right]}{2},$$

$$\widehat{\phi_1}(\xi)\left(\frac{1}{2}\int_{-at}^{at}e^{i|\xi|\theta}\,d\theta\right)=\frac{1}{2}\int_{-at}^{at}\mathcal{F}\left[\delta(x+\theta)*\phi_1(x)\right]d\theta,$$

$$\widehat{\phi_2}(\xi)\left(-\frac{1}{2}\int_0^{at}\int_0^{y}e^{i|\xi|u}\,du\,dy-\frac{1}{2}\int_0^{at}\int_0^{y}e^{-i|\xi|u}\,du\,dy\right)$$

$$=-\frac{1}{2}\Big(\int_0^{at}\int_0^{y}\mathcal{F}\left[\delta(x+u)*\phi_2(x)\right]du\,dy+\int_0^{at}\int_0^{y}\mathcal{F}\left[\delta(x-u)*\phi_2(x)\right]du\,dy\Big),$$

$$\widehat{\phi_3}(\xi)\left(\int_{-at}^{at}\int_0^{y}\int_0^{\tau}e^{i|\xi|u}\,du\,d\tau\,dy\right)=\int_{-at}^{at}\int_0^{y}\int_0^{\tau}\mathcal{F}\left[\delta(x+u)*\phi_3(x)\right]du\,d\tau\,dy.$$

Substituting the above identities into the formula (2.4), we obtain that

$$\begin{aligned}
\widehat{u}(\xi,t)=&-\frac{b^2}{(a^2-b^2)}\frac{\mathcal{F}\left[\delta(x+at)*\phi_0(x)\right]+\mathcal{F}\left[\delta(x-at)*\phi_0(x)\right]}{2}\\
&+\frac{a^2}{(a^2-b^2)}\frac{\mathcal{F}\left[\delta(x+bt)*\phi_0(x)\right]+\mathcal{F}\left[\delta(x-bt)*\phi_0(x)\right]}{2}\\
&-\frac{b^2}{(a^3-ab^2)}\frac{1}{2}\int_{-at}^{at}\mathcal{F}\left[\delta(x+\theta)*\phi_1(x)\right]d\theta\\
&+\frac{a^2}{(a^2b-b^3)}\frac{1}{2}\int_{-bt}^{bt}\mathcal{F}\left[\delta(x+\theta)*\phi_1(x)\right]d\theta\\
&-\frac{1}{(a^2-b^2)}\left(-\frac{1}{2}\int_0^{at}\int_0^{y}\mathcal{F}\left[\delta(x+u)*\phi_2(x)\right]du\,dy\right.\\
&\left.-\frac{1}{2}\int_0^{at}\int_0^{y}\mathcal{F}\left[\delta(x-u)*\phi_2(x)\right]du\,dy\right)\\
&+\frac{1}{(a^2-b^2)}\left(-\frac{1}{2}\int_0^{bt}\int_0^{y}\mathcal{F}\left[\delta(x+u)*\phi_2(x)\right]du\,dy\right.\\
&\left.-\frac{1}{2}\int_0^{bt}\int_0^{y}\mathcal{F}\left[\delta(x-u)*\phi_2(x)\right]du\,dy\right)\\
&+\frac{1}{(a^3-ab^2)}\frac{1}{2}\int_{-at}^{at}\int_0^{y}\int_0^{\tau}\mathcal{F}\left[\delta(x+u)*\phi_3(x)\right]du\,d\tau\,dy\\
&-\frac{1}{(a^2b-b^3)}\frac{1}{2}\int_{-bt}^{bt}\int_0^{y}\int_0^{\tau}\mathcal{F}\left[\delta(x+u)*\phi_3(x)\right]du\,d\tau\,dy.
\end{aligned}$$

Consequently, we get the inverse formula of $\widehat{u}$ given by

$$u(x,t)=-\frac{b^2}{(a^2-b^2)}\frac{\left(\delta(x+at)*\phi_0(x)\right)+\left(\delta(x-at)*\phi_0(x)\right)}{2}$$

$$
\begin{aligned}
&+ \frac{a^2}{(a^2-b^2)} \frac{(\delta(x+bt) * \phi_0(x)) + (\delta(x-bt) * \phi_0(x))}{2} \\
&- \frac{b^2}{(a^3-ab^2)} \frac{1}{2} \int_{-at}^{at} (\delta(x+\theta) * \phi_1(x))\, d\theta \\
&+ \frac{a^2}{(a^2b-b^3)} \frac{1}{2} \int_{-bt}^{bt} (\delta(x+\theta) * \phi_1(x))\, d\theta \\
&- \frac{1}{(a^2-b^2)} \left(-\frac{1}{2} \int_0^{at} \int_0^{y} (\delta(x+u) * \phi_2(x))\, du\, dy \right. \\
&\left. - \frac{1}{2} \int_0^{at} \int_0^{y} (\delta(x-u) * \phi_2(x))\, du\, dy \right) \\
&+ \frac{1}{(a^2-b^2)} \left(-\frac{1}{2} \int_0^{bt} \int_0^{y} (\delta(x+u) * \phi_2(x))\, du\, dy \right. \\
&\left. - \frac{1}{2} \int_0^{bt} \int_0^{y} (\delta(x-u) * \phi_2(x))\, du\, dy \right) \\
&+ \frac{1}{(a^3-ab^2)} \frac{1}{2} \int_{-at}^{at} \int_0^{y} \int_0^{\tau} (\delta(x+u) * \phi_3(x))\, du\, d\tau\, dy \\
&- \frac{1}{(a^2b-b^3)} \frac{1}{2} \int_{-bt}^{bt} \int_0^{y} \int_0^{\tau} (\delta(x+u) * \phi_3(x))\, du\, d\tau\, dy.
\end{aligned}
$$

The last formula is equivalent to the one given at (2.5), so the theorem is proved. □

For the generalized case, we will use the following result:

Lemma 1. *For an odd number $n \geq 3$, $m = \dfrac{n-3}{2}$ and $0 \leq k \leq m$ then*

$$
\int_{-R}^{R} e^{is|\xi|}(R^2 - s^2)^{m-k}\, ds = \frac{1}{2^k k!} \left(\frac{1}{R} \frac{\partial}{\partial R} \right)^k \left(\frac{1}{\omega_{n-1} R} \int_{\partial B(0,R)} e^{-ix.\xi}\, d\sigma(x) \right).
$$

For the proof, we refer the reader to Torchinsky's paper in [12]. Note that, in the case $k = m$, it follows that

$$
\begin{aligned}
\frac{\sin(a|\xi|t)}{|\xi|} &= \frac{1}{2} \int_{-at}^{at} e^{is|\xi|}\, ds \\
&= \frac{1}{2^{m+1} m!} \left(\frac{1}{a^2 t} \frac{\partial}{\partial t} \right)^m \left(\frac{1}{\omega_{n-1} at} \int_{\partial B(0,at)} e^{-ix.\xi}\, d\sigma(x) \right). \qquad (2.6)
\end{aligned}
$$

Differentiating with respect to t, then

$$
\cos(a|\xi|t) = \frac{1}{2^{m+1} m! a} \frac{\partial}{\partial t} \left(\frac{1}{a^2 t} \frac{\partial}{\partial t} \right)^m \left(\frac{1}{\omega_{n-1} at} \int_{\partial B(0,at)} e^{-ix.\xi}\, d\sigma(x) \right). \qquad (2.7)
$$

On the other hand, we have

$$\frac{\cos(b|\xi|t)}{|\xi|^2} - \frac{\cos(a|\xi|t)}{|\xi|^2} = \int_{bt}^{at} \frac{\sin(s|\xi|)}{|\xi|}\, ds = \frac{1}{2}\int_{bt}^{at}\int_{-s}^{s} e^{i\tau|\xi|}\, d\tau\, ds$$
$$= \int_{bt}^{at} \frac{1}{2^{m+1}m!}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^m \left(\frac{1}{\omega_{n-1}s}\int_{\partial B(0,s)} e^{-ix.\xi}\, d\sigma(x)\right) ds. \tag{2.8}$$

Integrating the above formula with respect to t, it implies that

$$\frac{1}{b}\frac{\sin(b|\xi|t)}{|\xi|^3} - \frac{1}{a}\frac{\sin(a|\xi|t)}{|\xi|^3} = \frac{1}{2}\int_0^t\int_{b\nu}^{a\nu}\int_{-s}^{s} e^{i\tau|\xi|}\, d\tau\, ds\, d\nu$$
$$= \int_0^t\int_{b\nu}^{a\nu} \frac{1}{2^{m+1}m!}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^m \left(\frac{1}{\omega_{n-1}s}\int_{\partial B(0,s)} e^{-ix.\xi}\, d\sigma(x)\right) ds\, d\nu. \tag{2.9}$$

Moreover, for each function $\theta \in \mathcal{S}(\mathbb{R}^n)$, we also have

$$\int_{\mathbb{R}^n} \frac{\sin(a|\xi|t)}{|\xi|}\theta(\xi)\, d\xi = \frac{1}{2^{m+1}m!}\left(\frac{1}{a^2t}\frac{\partial}{\partial t}\right)^m \frac{1}{\omega_{n-1}at}$$
$$\times \int_{\partial B(0,at)}\int_{\mathbb{R}^n} e^{-ix.\xi}\theta(\xi)\, d\xi\, d\sigma(x) = \frac{1}{2^{m+1}m!}\left(\frac{1}{a^2t}\frac{\partial}{\partial t}\right)^m \frac{1}{\omega_{n-1}at}\int_{\partial B(0,at)} \widehat{\theta}(x)\, d\sigma(x).$$

Therefore, we conclude that

$$\frac{1}{2^{m+1}m!}\left(\frac{1}{a^2t}\frac{\partial}{\partial t}\right)^m \widehat{\left(\frac{1}{\omega_{n-1}at}\int_{\partial B(0,at)} d\sigma(x)\right)}(\xi) = \frac{\sin(a|\xi|t)}{|\xi|}.$$

By the Fourier inversion and convolution formulas, we obtain the identity

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{\phi_1}(\xi)\frac{\sin(a|\xi|t)}{|\xi|}e^{i\xi.x}\, d\xi$$
$$= \frac{1}{2^{m+1}m!}\left(\frac{1}{a^2t}\frac{\partial}{\partial t}\right)^m \left(\frac{1}{\omega_{n-1}at}\int_{\partial B(x,at)} \phi_1(y)\, d\sigma(y)\right)$$
$$= \frac{\omega_n}{2^{m+1}m!\omega_{n-1}}\left(\frac{1}{a^2t}\frac{\partial}{\partial t}\right)^m \left((at)^{n-2}M_{at}(\phi_1)(x)\right)$$
$$= \frac{1}{(n-2)!!}\left(\frac{1}{a^2t}\frac{\partial}{\partial t}\right)^m \left((at)^{n-2}M_{at}(\phi_1)(x)\right).$$

Applying the same way for the expressions (2.7)–(2.9) and substituting the obtained identities into the formula (2.4), we have found an exact solution to the n-dimensional biwave equation, where $n \geq 3$ is an odd number:

Theorem 2. *The Cauchy initial value problem for the homogeneous n-dimensional biwave equation, where $n \geq 3$ is an odd number, has the following solution*

$$u(x,t) = \frac{1}{(n-2)!!(a^2-b^2)}\left[\frac{a^2}{b}\frac{\partial}{\partial t}\left(\frac{1}{b^2t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\big((bt)^{n-2}M_{bt}(\phi_0)(x)\big)\right.$$
$$-\frac{b^2}{a}\frac{\partial}{\partial t}\left(\frac{1}{a^2t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\big((at)^{n-2}M_{at}(\phi_0)(x)\big) + \frac{a^2}{b}\left(\frac{1}{b^2t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}$$
$$\times\big((bt)^{n-2}M_{bt}(\phi_1)(x)\big) - \frac{b^2}{a}\left(\frac{1}{a^2t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\big((at)^{n-2}M_{at}(\phi_1)(x)\big)$$
$$+\int_{bt}^{at}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{\frac{n-3}{2}}\big((s)^{n-2}M_s(\phi_2)(x)\big)\,ds$$
$$\left.+\int_0^t\int_{b\nu}^{a\nu}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{\frac{n-3}{2}}\big((s)^{n-2}M_s(\phi_3)(x)\big)\,ds\,d\nu\right].$$

Now we consider the case when n is an even number. The Hadamard's method of descent (see e.g. [4]) is useful to connect with the case in the odd dimensional space $\mathbb{R}^{n+1}$. For fixed $T > 0$, we choose a Schwartz function $\eta \in \mathcal{S}(\mathbb{R})$, such that $\eta(x_{n+1}) = 1$ for all $|x_{n+1}| \leq nT$. Let us denote

$$\overline{\phi_i}(x_1,x_2,\dots,x_n,x_{n+1}) = \phi_i(x_1,x_2,\dots,x_n)\eta(x_{n+1}), \quad i = 0,1,2,3.$$

It is easy to see that $\overline{\phi_i} \in \mathcal{S}(\mathbb{R}^{n+1})$. For $|x_{n+1}| \leq T$, $t \leq T$, the solution $\overline{u}(x_1,x_2,\dots,x_{n+1},t)$ to the Cauchy problem for $(n+1)$-dimensional biwave equation with initial valued functions $\overline{\phi_i}$, $i = 0,1,2,3$ does not depend on x_{n+1}. In particular,

$$u(x_1,x_2,\dots,x_n,t) = \overline{u}(x_1,x_2,\dots,x_n,0,t)$$

is the solution to the n-dimensional wave equation for all $|t| \leq T$. Since T is arbitrary, so u is the solution to the Cauchy problem in even dimensional space $\mathbb{R}^n$.

Lemma 2. *Given a function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, which does not depend on the last variable, i.e. $f(x_1,x_2,\dots,x_{n+1}) = g(x_1,x_2,\dots,x_n)$, then*

$$\mathcal{M}_t(f)(x,0) = \frac{2}{\omega_{n+1}}\int\limits_{B_n(0,1)}\frac{g(x+tz)}{\sqrt{1-|z|^2}}\,dz.$$

Proof. Observe that, for $\widetilde{x} = (x,0)$ and $\widetilde{y} = (y,y_{n+1})$, we have

$$\mathcal{M}_t(f)(\widetilde{x}) = \frac{1}{\omega_{n+1}}\int\limits_{\partial B_{n+1}(0,1)} f(\widetilde{x}+t\widetilde{y})\,d\sigma(\widetilde{y}).$$

We use the spherical coordinates given by

$$\begin{cases} y_1 = \sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-2}\sin\varphi_{n-1}\sin\varphi_n,\\ y_2 = \sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-2}\sin\varphi_{n-1}\cos\varphi_n,\end{cases}$$

$$\begin{cases} y_3 = \sin\varphi_1 \sin\varphi_2 \cdots \sin\varphi_{n-2} \cos\varphi_{n-1} \cos\varphi_n, \\ \cdots \\ y_n = \sin\varphi_1 \cos\varphi_2, \quad y_{n+1} = \cos\varphi_1, \end{cases}$$

where $0 \le \varphi_k \le \pi$, $k = 1, 2, \dots, n-1$ and $0 \le \varphi_n \le 2\pi$. The Jacobian of this transformation is calculated as

$$J = \sin^{n-1}\varphi_1 \sin^{n-2}\varphi_2 \cdots \sin\varphi_{n-1}.$$

Therefore

$$\mathcal{M}_t(f)(x,0) = \frac{1}{\omega_{n+1}} \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} g(x+ty) J \, d\varphi_1 \, d\varphi_2 \dots d\varphi_n.$$

Let us give $r = \sin\varphi_1$, and

$$\begin{cases} z_1 = r \sin\varphi_2 \cdots \sin\varphi_{n-1} \sin\varphi_n, \\ z_2 = r \sin\varphi_2 \cdots \sin\varphi_{n-1} \cos\varphi_n, \\ \cdots \\ z_n = r \cos\varphi_2. \end{cases}$$

The Jacobian of above transformation is calculated by the formula

$$J' = \frac{1}{r^{n-1} \sin^{n-2}\varphi_2 \sin^{n-3}\varphi_3 \cdots \sin\varphi_{n-1}}.$$

Finally, we obtain that

$$\begin{aligned} \mathcal{M}_t(f)(x,0) &= \frac{2}{\omega_{n+1}} \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \int_0^1 g(x+tz) \frac{1}{\cos\varphi_1} J \, dr \, d\varphi_2 \dots d\varphi_n \\ &= \frac{2}{\omega_{n+1}} \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \int_0^1 g(x+tz) \frac{1}{\sqrt{1-|z|^2}} J \, dr \, d\varphi_2 \dots d\varphi_n \\ &= \frac{2}{\omega_{n+1}} \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \int_0^1 g(x+tz) \frac{1}{\sqrt{1-|z|^2}} J J' \, dz_1 \, dz_2 \dots dz_n \\ &= \frac{2}{\omega_{n+1}} \int_{B_n(0,1)} \frac{g(x+tz)}{\sqrt{1-|z|^2}} \, dz. \end{aligned}$$

So the lemma is proved. □

We use the notation $\widetilde{\mathcal{M}}_t(f)(x) = \frac{2}{\omega_{n+1}} \int_{B_n(0,1)} \frac{f(x+tz)}{\sqrt{1-|z|^2}} \, dz$ for a modified spherical mean of f (see e.g. [9, 11]). Applying the result of Lemma 2, we obtain the formula of the solution to the biwave equation in the even dimensional space $\mathbb{R}^n$:

Theorem 3. *The Cauchy initial value problem for the homogeneous n-dimensional biwave equation, where $n \ge 2$ is an even number, has the following solution*

$$\begin{aligned} u(x,t) = \frac{1}{(n-1)!!(a^2-b^2)} &\left[\frac{a^2}{b} \frac{\partial}{\partial t} \left(\frac{1}{b^2 t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left((bt)^{n-1} \widetilde{\mathcal{M}}_{bt}(\phi_0)(x) \right) \right. \\ &- \frac{b^2}{a} \frac{\partial}{\partial t} \left(\frac{1}{a^2 t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left((at)^{n-1} \widetilde{\mathcal{M}}_{at}(\phi_0)(x) \right) \end{aligned}$$

$$+\frac{a^2}{b}\left(\frac{1}{b^2 t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left((bt)^{n-1}\widetilde{\mathcal{M}}_{bt}(\phi_1)(x)\right)-\frac{b^2}{a}\left(\frac{1}{a^2 t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}$$
$$\times\left((at)^{n-1}\widetilde{\mathcal{M}}_{at}(\phi_1)(x)\right)+\int_{bt}^{at}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{\frac{n-2}{2}}\left((s)^{n-1}\widetilde{\mathcal{M}}_{s}(\phi_2)(x)\right)ds$$
$$-\int_0^t\int_{b\nu}^{a\nu}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{\frac{n-2}{2}}\left((s)^{n-1}\widetilde{\mathcal{M}}_{s}(\phi_3)(x)\right)ds\,d\nu\Bigg].$$

By a similar idea with the Duhamel principle for wave equations, the solution of the Cauchy problem for the nonhomogeneous biwave equation will be given at the next theorem

Theorem 4. *The solution of the equation* (1.1)–(1.2) *takes the form* $u=\widetilde{u}+v$, *where* $\widetilde{u}$ *is the solution of the equation* (2.1)–(2.2) *and*

$$v\left(x,t\right)=\int_0^t\omega\left(x,t,\tau\right)d\tau,$$

where $\omega\left(x,t,\tau\right)$ *is the solution of the homogeneous biwave equation*

$$\left(\frac{\partial^2}{\partial t^2}-a^2\Delta\right)\left(\frac{\partial^2\omega}{\partial t^2}-b^2\Delta\omega\right)=0,\quad t>\tau,$$

with the initial conditions

$$\omega|_{t=\tau}=0,\quad \left.\frac{\partial\omega}{\partial t}\right|_{t=\tau}=0,\quad \left.\frac{\partial^2\omega}{\partial t^2}\right|_{t=\tau}=0,\quad \left.\frac{\partial^3\omega}{\partial t^3}\right|_{t=\tau}=f\left(x,\tau\right).$$

Proof. We start with the observation that

$$\frac{\partial^4 v}{\partial t^4}-\left(a^2+b^2\right)\frac{\partial^2}{\partial t^2}\Delta v+a^2b^2\Delta^2 v=f\left(x,t\right)$$
$$+\int_0^t\left(\frac{\partial^4\omega}{\partial t^4}-\left(a^2+b^2\right)\frac{\partial^2}{\partial t^2}\Delta\omega+a^2b^2\Delta^2\omega\right)d\tau=f\left(x,t\right).$$

Then, the above identity follows that

$$\frac{\partial^4 u}{\partial t^4}-\left(a^2+b^2\right)\frac{\partial^2}{\partial t^2}\Delta u+a^2b^2\Delta^2 u=f\left(x,t\right).$$

Moreover,

$$u|_{t=0}=\widetilde{u}|_{t=0}+v|_{t=0}=\phi_0\left(x\right)+0=\phi_0\left(x\right),$$
$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=\left.\frac{\partial\widetilde{u}}{\partial t}\right|_{t=0}+\left.\frac{\partial v}{\partial t}\right|_{t=0}=\phi_1\left(x\right)+0=\phi_1\left(x\right),$$
$$\left.\frac{\partial^2 u}{\partial t^2}\right|_{t=0}=\left.\frac{\partial^2\widetilde{u}}{\partial t^2}\right|_{t=0}+\left.\frac{\partial^2 v}{\partial t^2}\right|_{t=0}=\phi_2\left(x\right)+0=\phi_2\left(x\right),$$
$$\left.\frac{\partial^3 u}{\partial t^3}\right|_{t=0}=\left.\frac{\partial^3\widetilde{u}}{\partial t^3}\right|_{t=0}+\left.\frac{\partial^3 v}{\partial t^3}\right|_{t=0}=\phi_3\left(x\right)+0=\phi_3\left(x\right).$$

So the theorem is proved. □

3 Example

Let us give an example demonstrating Theorem 1. Consider the equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial t^2} - \frac{1}{4}\frac{\partial^2 u}{\partial x^2}\right) = 0 \tag{3.1}$$

with the initial conditions

$$u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin x, \quad \left.\frac{\partial^2 u}{\partial t^2}\right|_{t=0} = \cos x, \quad \left.\frac{\partial^3 u}{\partial t^3}\right|_{t=0} = 0. \tag{3.2}$$

The solution of the equation (3.1)–(3.2) is given by the formula

$$u(x,t) = \frac{1}{3}\left(4\cos\left(\frac{t}{2}\right)\cos x - 4\cos t\cos x - \left(-8\sin\left(\frac{t}{2}\right) + \sin t\right)\sin x\right).$$

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