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Practical Error Analysis for the Three-Level Bilinear FEM and Finite-Difference Scheme for the 1D Wave Equation With Non-Smooth Data

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Abstract. We deal with the standard three-level bilinear FEM and finite-difference scheme with a weight to solve the initial-boundary value problem for the 1D wave equation. We consider the rich collection of initial data and the free term which are the Dirac δ-functions, discontinuous, continuous but with discontinuous derivatives and from the Sobolev spaces, accomplish the practical error analysis in the L^2, L^1, energy and uniform norms as the mesh refines and compare results with known theoretical error bounds.

Keywords: 1D wave equation, non-smooth data, bilinear FEM, finite-difference scheme, practical error analysis.

AMS Subject Classification: 65M60; 65M06; 65M15.

1 Introduction

We deal with the initial-boundary value problem for the 1D wave equation and discuss error behavior for the standard three-level regularized bilinear finite element method (FEM) and finite-difference scheme with a weight. In [10] (where references to a lot of related papers can be also found) general error bounds for three- and two-level FEMs were proved in several norms in continuous dependence on the order of smoothness of data, i.e. two initial functions in the Sobolev/Nikolskii spaces and the free term in the equation in the spaces of the dominating mixed smoothness in (x,t) (much broader then the Sobolev or Nikolskii spaces of the same order). The negative order of smoothness is taken into account thus one of the initial functions and the free term can be distributions like the Dirac δ-functions.

More precisely, the error bounds in L^2, energy and (in 1D) uniform spatial norms and uniformly in time of the orders respectively

$$O\big((\tau+h)^{2\alpha/3}\big),\ 0\le\alpha\le 3;\ \ O\big((\tau+h)^{2(\alpha-1)/3}\big),\ 1\le\alpha\le 4$$

and $O\big((\tau+h)^{2(\alpha-1/2)/3}\big)$, $1/2<\alpha<7/2$, were derived. For half-integer α, these bounds cover such practically interesting cases of non-smooth data as the Dirac δ-functions, discontinuous or continuous but with discontinuous derivatives functions, etc.

These error orders differ significantly from those in the elliptic and parabolic cases. Nevertheless the sharpness of the error bounds in the L^2 and energy norms was confirmed in [8, 9] by the lower error bounds in the corresponding spaces, for each of two initial functions and the free term. Notice that this was accomplished on sequences of rapidly oscillating elements of the spaces but not for their specific typical elements. Also up to now no similar lower bounds in the uniform norm are known.

In addition, obviously the L^2 and energy error bounds (the latter also involves the L^2 spatial norms) are weakened after replacing the L^2 norms by L^1 ones. Nevertheless, according to [9] these weakened bounds remain sharp in the same data spaces.

From the practical point of view, it is essential to know whether the discussed error bounds are sharp for typical elements of the above mentioned non-smooth data spaces. At the moment there exists no any theoretical answer to this question; it seems that a very delicate asymptotic analysis of the error behavior is required to get such answer.

To understand at least a practical answer (that is also important by itself), in this paper we analyze practical error orders as mesh refines for the rich collection of namely such the typical non-smooth two initial functions and free term (separately for each of them). For the initial functions, we also treat the case of integer α, i.e. functions from the Sobolev spaces. We also include examples with mismatches between the initial and boundary data.

The main result consists in finding out that the discussed error orders in the L^2, energy as well as uniform spatial norms are actually sharp with the high precision for all the considered typical data; in particular, it is observed how sensitively the orders increase when the data smoothness grows. On the other hand, we reliably observe higher error orders in the weaker L^1 and $W^{1,1}$ spatial norms; thus for these norms the error orders in the spaces of data and for their typical elements differ significantly. The latter fact as well as the presented results in general could stimulate further studies.

2 The initial-boundary value problem, numerical methods and the theoretical error bounds

We deal with the initial-boundary value problem for the 1D wave equation

$$D_t^2 u - a^2 D_x^2 u = f(x,t) \quad \text{in} \quad Q := \Omega \times S, \tag{2.1}$$

$$u|_{t=0} = u_0(x), \quad D_t u|_{t=0} = u_1(x), \quad x \in \Omega := (-X/2, X/2), \tag{2.2}$$

$$u|_{x=-X/2} = g_0(t), \quad u|_{x=X/2} = g_1(t), \quad t \in S := (0,T). \tag{2.3}$$

Hereafter D_t and D_x are the weak partial derivatives in t and x; also $a =$ const > 0. Recall that in dependence on regularity of the data u_0, u_1 and f (as well as g_0 and g_1), there exists a unique weak solution from the energy class, or even weaker (possibly discontinuous), or strong, or classical solution to this IBVP, in particular, see [2,10]. Below we treat all these four types of solutions.

Recall that *the weak solution from the energy class* $u \in C(\bar{S}, H^1(\Omega))$ with $D_t u \in C(\bar{S}, L^2(\Omega))$ satisfies the integral identity

$$\int_Q \big(-(D_t u) D_t \eta + a^2 (D_x u) D_x \eta \big) dx dt = \int_\Omega u_1 \eta|_{t=0}\, dx + \int_Q f\eta\, dx dt \tag{2.4}$$

for any $\eta \in L^1(S, H_0^1(\Omega))$ with $D_t\eta \in L^1(S, L^2(\Omega))$ and $\eta|_{t=T} = 0$, and the initial-boundary conditions $u|_{t=0} = u_0$ and (2.3) in the classical sense (since in our 1D case $u \in C(\bar{Q})$). For the free term-distribution $f = D_t g$ with $g|_{t=0} = 0$, the last term on the right should be understood as $-\langle g, D_t \eta\rangle$.

The weaker solution $u \in C(\bar{S}, L^2(\Omega))$ with $I_t u \in C(\bar{S}, H^1(\Omega))$ satisfies another integral identity

$$\int_Q \big(-u D_t \eta + a^2 (D_x I_t u) D_x \eta \big) dx dt = \int_\Omega u_0 \eta|_{t=0}\, dx$$
$$+ \big\langle u_1, (I_t^* \eta)|_{t=0} \big\rangle + \big\langle f, I_t^* \eta \big\rangle$$

for the same η as above and the boundary conditions $I_t u|_{x=-X/2} = I_t g_0$ and $I_t u|_{x=X/2} = I_t g_1$ in the classical sense. Here $I_t u(x,t) := \int_0^t u(x,\theta)\, d\theta$ is a primitive in t function for u, $I_t^* \eta(x,t) := \int_t^T \eta(x,\theta)\, d\theta$ as well as u_1 and f are distributions respectively on Ω and Q.

The strong solution $u \in C(\bar{S}, H^2(\Omega))$ with $D_t u \in C(\bar{S}, H^1(\Omega))$ and $D_t^2 u \in C(\bar{S}, L^2(\Omega))$ satisfies equation (2.1) in $C(\bar{S}, L^2(\Omega))$ and the initial-boundary conditions (2.2)–(2.3) in the classical sense.

Notice that for the IBVP (2.1)–(2.3) all the above mentioned types of solutions can be represented by one and the same D'Alembert-type formula so we could omit their definitions.

Now we present numerical methods to solve the IBVP (2.1)–(2.3) which error we analyze below. Let $\bar{\omega}_h = \{x_i = -X/2 + ih;\ 0 \le i \le n\}$ and $\bar{\omega}^\tau = \{t_m = m\tau;\ 0 \le m \le M\}$ be the uniform meshes (for simplicity) on $\bar{\Omega}$ and $\bar{S}$, with the steps $h = X/n$ and $\tau = T/M$. Define also the internal mesh $\omega_h = \bar{\omega}_h \setminus \{\pm X/2\}$ and the mesh norms $\|\varphi\|_{C_h} = \max_{\bar{\omega}_h} |\varphi_i|$ and $\|\varphi\|_{L^p(\omega_h)} = \big(h \sum_{i=1}^{n-1} |\varphi_i|^p \big)^{1/p}$, $p \ge 1$.

Let $\mathcal{S}_h$ be the FEM space of functions in $C(\bar{\Omega})$ which are linear over each element $[x_{i-1}, x_i]$, $1 \le i \le n$. Let $\mathcal{S}^\tau$ be the similar FEM space associated to $\bar{S}$ and $\bar{\omega}^\tau$. For $w \in C(\bar{\Omega})$, let $\widehat{w} \in \mathcal{S}_h$ be its interpolant such that $w(x_i) = \widehat{w}(x_i)$, $0 \le i \le n$.

We introduce the mesh operators in x and t

$$B_h v_i = \frac{1}{6} v_{i-1} + \frac{2}{3} v_i + \frac{1}{6} v_{i+1}, \quad L_h v_i = -\frac{v_{i-1} - 2v_i + v_{i+1}}{h^2}, \ 1 \le i \le n-1,$$
$$\partial_t y^m = \frac{y^{m+1} - y^m}{\tau}, \ 0 \le m \le M-1; \ \ \bar{\partial}_t y^m = \frac{y^m - y^{m-1}}{\tau}, \ 1 \le m \le M;$$

let also $\bar{\partial}_t y^0 = 0$. Clearly B_h and L_h are the scaled mass and stiffness operators corresponding to the FEM space $\mathcal{S}_h$.

Let the approximate solution $v \in \mathcal{S}_h \otimes \mathcal{S}^\tau$. We study *the regularized bilinear FEM* written in the following three-level in time operator form [10]

$$(B_h + \sigma\tau^2 L_h)\partial_t \bar{\partial}_t v^m + L_h v^m = f^{h,\tau,m} \ \text{ on } \ \omega_h, \ 1 \le m \le M-1, \tag{2.5}$$
$$(B_h + \sigma\tau^2 L_h)\partial_t v_0^1 + \frac{\tau}{2} L_h v^0 = u_1^h + \frac{\tau}{2} f_0^{h,\tau,0} \ \text{ on } \ \omega_h, \tag{2.6}$$
$$v^0 = v^0_{\sigma_0} \ \text{ or } \ v^0 = \widehat{u}_0 \ \text{ on } \ \bar{\omega}_h, \tag{2.7}$$
$$v_0^m = g_0(t_m), \ \ v_n^m = g_1(t_m), \ \ 1 \le m \le M, \tag{2.8}$$

with the regularizing parameter $\sigma \ge 1/4$ (for simplicity). Here $v^0_{\sigma_0}$ satisfies

$$(B_h + \sigma_0\tau^2 L_h) v^0_{\sigma_0} = u_0^h \ \text{ on } \ \omega_h, \ \ v^0_{\sigma_0\, i} = u^h_{0i}, \ \ i = 0, n, \tag{2.9}$$

with the parameter $\sigma_0 \ge \sigma - 1/4$. We omit the original Galerkin form of the FEM based on the regularized integral identity (2.4), see [10]. Here for $f \in L^1(Q)$, $w \in L^1(\Omega)$ and $z \in L^1(S)$ the following FEM averages are utilized

$$f_i^{h,\tau,m} = (f_i^h)^{\tau,m}, \ \ w_i^h = \frac{1}{h}\int_{x_{i-1}}^{x_{i+1}} w(x) e_i^h(x)\, dx, \ 1 \le i \le n-1, \tag{2.10}$$
$$z^{\tau,0} = \frac{2}{\tau}\int_0^\tau z(t) e^{\tau,0}(t)\, dt, \ z^{\tau,m} = \frac{1}{\tau}\int_{t_{m-1}}^{t_{m+1}} z(t) e^{\tau,m}(t)\, dt, \ 1 \le m \le M-1, \tag{2.11}$$

where $e_i^h(x) = \max\{1 - |x/h - i|, 0\}$ and $e^{\tau,m}(t) = \max\{1 - |t/\tau - m|, 0\}$ are the well-known "hat" functions. The formula (2.10) can be used for $i = 0, n$ as well using the formulas $x_{-1} = -X/2 - h$ and $x_{n+1} = X/2 + h$ and, say, the even or odd extension of w with respect to $\pm X/2$ outside Ω. Below in our computations we use these averages for discontinuous f, w (including $w = u_0, u_1$) and z.

In more general situation where f, w and z are distributions respectively on Q, Ω and S, we set

$$f_i^{h,\tau,m} = \langle f, e_i^h e^{\tau,m}\rangle, \ \ w_i^h = \langle w, e_i^h\rangle, \ \ z^{\tau,m} = \langle z, e^{\tau,m}\rangle. \tag{2.12}$$

Below we need them for f, w (including u_1) and z like the Dirac δ-functions.

For the Hölder-continuous $w = u_0$ and u_1 below in our computations we often take simply $w_i^h = w(x_i)$ that is most usual in practice.

Below in computations we consider f only with the separated variables, i.e. $f(x,t)=w(x)z(t)$, and compute $f^{h,\tau}=w^h z^\tau$ accordingly to the type of w and z (i.e., distributions or discontinuous functions).

After simplifying the mass operator B_h down to the unit one I (the mass lumping procedure) and taking $\sigma_0=0$, we get the well-known three-level symmetric in space and time *finite-difference method* (FDM), or scheme, with the weight σ [3] that we also analyze below.

Next we need to recall the basic error bounds for the above FEM which are the simplest cases of general error bounds *in terms of data smoothness* from [10]. Define the uniform norm $\|r\|_{C_\tau(\bar{S};\mathcal{B})}=\max_{\bar{\omega}^\tau}\|r(t_m)\|_{\mathcal{B}}$ in the space of functions on $\bar{\omega}^\tau$ with values in a Banach space $\mathcal{B}$. Let $\varepsilon_0>0$ and first $g_0=g_1=0$.

(i) For $\sigma\geq 1/4$ and $v_0=v^0_{\sigma_0}$, the following error bound in the extended $L^2(\Omega)$ norm holds

$$\begin{aligned}&\|u-v\|_{C_\tau(\bar{S};L^2(\Omega))}+\|D_x(\widehat{I_t u}-I_t v)\|_{C_\tau(\bar{S};L^2(\Omega))}\\&\quad\leq c(\tau+h)^{2\alpha/3}\big(\|u_0\|_{H^{(\alpha)}}+\|u_1\|_{H^{(\alpha-1)}}+\|f\|_{F^{(\alpha_1,\alpha_2)}}\big),\ 0\leq\alpha\leq 3.\end{aligned} \tag{2.13}$$

Hereafter c is independent of h and τ. For $1\leq\alpha\leq 3$, the bound is valid for simpler $v_0=\widehat{u}_0$ as well, and also one can replace $u-v$ by $\widehat{u}-v$ in it; both these moments are in use below.

The left-hand side of (2.13) contains the additional non-standard term with the time primitive $I_t u$; it plays an important role in some recent applications to optimal control problems [5].

(ii) For $\sigma\geq 1/4+\varepsilon_0$ and $v_0=\widehat{u}_0$, the following error bound in the energy-type norm holds

$$\begin{aligned}&\|D_x(\widehat{u}-v)\|_{C_\tau(\bar{S};L^2(\Omega))}+\|\bar{\partial}_t(\widehat{u}-v)\|_{C_\tau(\bar{S};L^2(\Omega))}\\&\quad\leq c(\tau+h)^{2(\alpha-1)/3}\big(\|u_0\|_{H^{(\alpha)}}+\|u_1\|_{H^{(\alpha-1)}}+\|f\|_{F^{(\alpha_1,\alpha_2)}}\big),\ 1\leq\alpha\leq 4.\end{aligned} \tag{2.14}$$

(iii) For $\sigma\geq 1/4+\varepsilon_0$ and $v_0=\widehat{u}_0$, the following error bound in the mesh uniform norm holds

$$\begin{aligned}\|u-v\|_{C_\tau(\bar{S};C_h)}&=\max_{\bar{\omega}_h\times\bar{\omega}^\tau}|(u-v)(x_i,t_m)|\leq c(\tau+h)^{2(\alpha-1/2)/3}\\&\times\big(\|u_0\|_{H^{(\alpha)}}+\|u_1\|_{H^{(\alpha-1)}}+\|f\|_{F^{(\alpha_1,\alpha_2)}}\big),\ \frac{1}{2}<\alpha<\frac{7}{2}.\end{aligned} \tag{2.15}$$

The mesh uniform norm is often especially valuable in practice. Recall that bound (2.15) is derived in [10] as a consequence of (2.13)–(2.14).

In addition, in the above error bounds $\alpha_1+\alpha_2=\alpha-1$ and the pair (α_1,α_2) belongs to some sets on the plane that we need not to reproduce in full generality and confine ourselves by some particular cases below.

Concerning the spaces $H^{(\alpha)}$ for the initial data in the listed error bounds, $H^{(0)}=L^2(\Omega)$ and, for integer $\alpha=1,2,3,4$, $H^{(\alpha)}$ are the subspaces of functions w in the Sobolev spaces $W^{\alpha,2}(\Omega)$ with $w|_{x=\pm X/2}=0$ and in addition $D_x^2 w|_{x=\pm X/2}=0$ for $\alpha=3,4$. For non-integer $0<\alpha<4$, $H^{(\alpha)}$ are similar subspaces in the Nikolskii spaces $H^{\alpha,2}(\Omega)$, see some details in [10]. Importantly for what follows, $H^{(1/2)}$ contains the space $BV(\bar{\Omega})$ of functions of bounded variation on $\bar{\Omega}$ which can be discontinuous, $H^{(3/2)}$ contains $w \in H^{(1)}$ with $D_x w \in BV(\bar{\Omega})$, etc. Finally, for negative $-1 < \alpha < 0$, $H^{(\alpha)}$ consists of distributions $w = D_x W$ such that $W \in H^{\alpha+1,2}(\Omega)$.

Concerning the spaces $F^{(\alpha_1,\alpha_2)}$ (for f) of the dominating mixed smoothness of order α_1 in x and order α_2 in t in the sense of the anisotropic Lebesgue space $L^{2,1}(Q)$, below we deal with the case $f(x,t) = w(x)z(t)$ and integer $\alpha_2 = 0, 1, 2$ only. For such α_2, in the above error bounds they can be even enlarged up to the spaces $\bar{F}^{(\alpha_1,\alpha_2)}$, and for the mentioned f the property $f \in \bar{F}^{(\alpha_1,\alpha_2)}$ means that, first, $w \in H^{(\alpha_1)}$ and, second:

(1) for $\alpha_2 = 0$, z is the distribution $z = D_t Z$ such that $Z \in BV(\bar{S})$, $Z(0^+) = 0$ and $\alpha_1 \in [-1, 2], [0, 3], (-1/2, 5/2)$ respectively in bounds (2.13)–(2.15);

(2) for $\alpha_2 = 1$, $z \in BV(\bar{S})$, $z(0^+) = 0$ and $\alpha_1 \in [0, 1], [-1, 2], [0, 3/2)$ respectively in bounds (2.13)–(2.15);

(3) for $\alpha_2 = 2$, $z \in C(\bar{S})$, $z(0) = 0$ with $D_t z \in BV(\bar{S})$, $D_t z(0^+) = 0$ and $\alpha_1 \in [0, 1], [0, 1/2)$ respectively in bounds (2.14)–(2.15).

Here the restrictions $z(0^+) = 0$, $z(0) = 0$ and $D_t z(0^+) = 0$ could be generalized but this is not required below.

The case of non-zero g_0 and g_1 is auxiliary in this paper, and we take them smooth only and mainly to simplify our choice of u_0 and u_1 below. Thus the rather simple 2nd order error bound in the energy-type norm (which is the strongest of all three norms considered above taking (2.13) with $\widehat{u} - v$ instead of $u - v$)

$$\|D_x(\widehat{u} - v)\|_{C_\tau(\bar{S};L^2(\Omega))} + \|\bar{\partial}_t(\widehat{u} - v)\|_{C_\tau(\bar{S};L^2(\Omega))} \le c(\tau + h)^2 \|u\|_{C^4(\bar{Q})}$$

is enough for our purposes since it cannot deteriorate the above error bounds (the auxiliary smooth solutions

$$u(x,t) = g_0(t)\big(1/2 - (x/X)\big) + g_1(t)\big((x/X) + 1/2\big)$$

having these boundary data can be taken into account). This bound follows from the proof in [10].

It seems that similar error bounds are valid in the case of the FDM. To derive them from the error bounds for the FEM, the techniques of reducing the former to the latter could be applied, in particular, see [6, 8]; some error bounds for the FDM can be found in [1].

The lower error bounds in the corresponding spaces of data u_0, u_1 and f (separately for each of them) of the same orders as in (2.13) and (2.14) are contained in [8] and in more general form in [9]. They were proved on sequences of rapidly oscillating elements in the spaces but not for their specific typical elements. But we emphasize that *no lower bounds* corresponding to bound (2.15) are known.

Below we analyze the most interesting cases of half-integer and integer values of $\alpha \in (0, 4)$. For convenience of comparing to practical error orders below, we put the corresponding orders of error bounds respectively (2.13), (2.15) and

(2.14) in Table 1 (non-positive values are replaced by dashes). Note that the orders are the same along "diagonals" of the arising rectangular matrix, and they increase by 1/3 when passing from one diagonal to the next to the right one (excluding the last "diagonal" consisting of a unique element).

Table 1. The theoretical error orders in dependence on α

α	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$
$\min\{2\alpha/3, 2\}$	0.333...	0.666...	1	1.333...	1.666...	2	2
$2(\alpha - 1/2)/3$	–	0.333...	0.666...	1	1.333...	1.666...	2
$2(\alpha - 1)/3$	–	–	0.333...	0.666...	1	1.333...	1.666...

3 Practical error analysis

We study the practical error behavior of the following errors

$$\|u - v\|_{C_\tau(\bar{S};L^p(\Omega))}, \quad \|u - v\|_{C_\tau(\bar{S};L^p(\omega_h))}, \quad \|D_x(\widehat{I_t u} - I_t v)\|_{C_\tau(\bar{S};L^2(\Omega))},$$
$$\|D_x(\widehat{u} - v)\|_{C_\tau(\bar{S};L^p(\Omega))}, \quad \|\bar{\partial}_t(\widehat{u} - v)\|_{C_\tau(\bar{S};L^2(\Omega))}, \quad \|u - v\|_{C_\tau(\bar{S};C_h)}$$

with $p = 1, 2$ treating the terms on the left-hand sides in bounds (2.13)–(2.14) separately. The main value is $p = 2$ since only it appears in bounds (2.13)–(2.14). Below in tables we mark shortly the listed errors respectively as

$$L^p, \ \ L^p_h, \ \ SW_h^{1,-1;2}, \ \ W_h^{1,0;p}, \ \ W_h^{0,1;2}, \ C_h.$$

The subscript h means that we use $\widehat{u}$ and $\widehat{I_t}u$ instead of u and $I_t u$ themselves (also recall that the $L^p(\Omega)$ and $L^p(\omega_h)$ norms are equivalent uniformly in h for $\varphi \in \mathcal{S}_h$ with $\varphi|_{x=\pm X/2} = 0$). The notation $SW_h^{1,-1;2}$, $W_h^{1,0;p}$ or $W_h^{0,1;2}$ conform respectively the dominating mixed smoothness of order 1 in x and -1 in t and the anisotropic Sobolev smoothness of order 1 in x or t only. The L^1 and L^1_h errors are often used in the case of discontinuous exact solutions in various applications including gas dynamics, for example, see [4]. In addition we utilize the $W_h^{1,0;1}$ error in the case of continuous piecewise smooth solutions; note that $\|D_x\widehat{w}\|_{L^1(\Omega)} = \sum_{i=1}^n |w_i - w_{i-1}|$ is the variation of w over $\bar{\omega}_h$.

Bounds (2.13)–(2.14) are clearly weakened after replacing $L^2(\Omega)$ by $L^1(\Omega)$. Nevertheless, according to [9] these weakened bounds remain sharp in the same data spaces. Remind that these lower error bounds were actually proved on sequences of rapidly oscillating elements in the spaces. On the contrary, in what follows we observe that for some typical elements of these spaces the practical error orders for $p = 1$ *are always of higher order* than for $p = 2$ (of course, when the latter are less than 2).

We use exact solutions u to the IBVP (2.1)–(2.3) constructed by the classical D'Alembert formula and the method of reflections but omit the arising rather elementary formulas for brevity.

We accomplished a number of preliminary numerical experiments and settled on the following unified strategy for computing practical error orders. We select the following values of n: $\{n_k\}_{k=1}^5 = \{1000, 1500, 2200, 3300, 5000\}$; note that $n_k/n_{k-1} = 1.5$ for $k = 2, 4$ and $n_k/n_{k-1} \approx 1.5$ for $k = 3, 5$. We prefer such choice compared to the most widespread one $n_k/n_{k-1} = 2$ since consider 2 being a too high ratio according to our experiments; see a similar choice previously in [7]. We also tried smaller values of n but restrict our presentation below by the above relatively large values n_k to achieve more reliability of practical error orders and their closeness to the corresponding theoretical ones (the observed convergence of the former to latter is not so fast especially for less smooth data).

We choose the simplest case of the square mesh with $\tau = h$. We also take $a = 1$ and $X = 1$ thus the characteristics of the 1D wave equation (2.1) starting from the points $\bar{\omega}_h \times \{0\}$ on the (x, t)-plane go through the mesh nodes. Looking ahead, this also means that the singularities of u (i.e., discontinuities of u or its derivatives) in all our examples are situated at the nodes of $\bar{\omega}_h$ for each time level in $\bar{\omega}^\tau$. We also take some $T \leq 0.5$ confining ourselves by the case without reflections of the characteristics from the boundary. It is known that in similar situations error orders can sometimes be improved. But we have found that this is not the case in our study and therefore restricted ourselves only by the square mesh as more complicated to confirm the sharpness of the error bounds.

Assuming the following asymptotic behavior of an error

$$r_n \sim c(\tau + h)^\gamma \quad \text{for} \quad h = X/n, \quad \tau = h$$

as $n \to \infty$ and considering $n = n_{k-1}, n_k$, we calculate the practical error orders according to the formula

$$\gamma_k = \ln \frac{r_{n_{k-1}}}{r_{n_k}} \Big/ \ln \frac{n_k}{n_{k-1}}$$

and expect that γ_k becomes closer and closer to γ as n_k grows.

In the rows of all tables below, we present errors r_n for $n = n_1 = 1000$ and then the practical error orders γ_k for $n_k = n_2, n_3, n_4, n_5$.

We have not found any essential differences between the results for the FEM and FDM, and below mainly the results for the FEM are given unless the opposite is explicitly stated. In addition, we take only $\sigma = 0.25$ or 0.5.

3.1 Practical error orders depending on the smoothness of $\mathbf{u_0}$

Our first collection of seven Examples A_α, $\alpha = 1/2, 3/2, 5/2, 7/2$ and $\alpha = 1, 2, 3$ is chosen to analyze practical error orders depending on the $L^2(\Omega)$-smoothness of order α of u_0. In them functions u_0 are nothing more than piecewise-power (or piecewise-linear for $\alpha = 3/2$) functions of order α. In addition in Example $A^m_{1/2}$ we consider the effect of a mismatch between smooth (constant) u_0 and the zero boundary data.

We assume that $u_1 = 0$ and $f = 0$ in all the examples. We mostly take $T = 0.4$, $\sigma = 0.5$ and set simply $v_0 = \widehat{u}_0$ unless the opposite is explicitly stated.

Example $\mathbf{A_{1/2}}$. We begin with $u_0 = H$ as the Heaviside step function, i.e. $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x > 0$. Obviously u_0 is piecewise-constant and discontinuous on Ω but it belongs to $H^{(1/2)}$. In addition, we use $g_0(t)=0$ and $g_1(t)\equiv 1$ to avoid a mismatch of the initial and boundary data. The exact weaker solution u is also piecewise-constant and discontinuous (on $\bar{Q}$).

Note that $u_{0i}^h=H(x_i)$, for all $i\neq n/2$ (we take these values for $i=0,n$ as well), and $u_{0n/2}^h=1/2$ (for even n). We set simply $v_0=u_0^h$. Table 2 contains results in Example $A_{1/2}$ for the FDM. The final practical orders differ from the theoretical ones within 1.7% for the terms of the error bound (2.13); hereafter all percents are relative with respect to the theoretical orders. Note that the L^2 and L^1 error orders oscillate while the $SW_h^{1,-1;2}$ one is decreasing. Also the L^1 error and orders are clearly better than the L^2 ones.

The same error results are valid also for $u_0(x)=(1/2)\operatorname{sgn} x-x$ and $g_0(t)=g_1(t)=0$ since the function $x+1/2$ solves both the IBVP (2.1)–(2.3) and the FEM (2.5), (2.6) and (2.8) together with the corresponding FDM simplification for $f=u_1=0$, $u_0(x)=x+1/2$, the above $g_0(t)=0$ and $g_1(t)\equiv 1$ and $v_0=u_0^h$ on $\bar{\omega}_h$.

There is no significant difference in results for the FEM and another mesh initial function $v_0=v_{\sigma_0}^0$ (see (2.9)) with $v_{\sigma_0 i}^0=H(x_i)$, $i=0,n$, for $\sigma_0=1/6$. In particular, the final practical orders are sequentially 0.328, 0.336 and 0.468; hereafter the corresponding full tables are omitted for brevity.

Example $\mathbf{A^m_{1/2}}$. We take the simplest $u_0(x)\equiv 1$ but in the mismatch with the zero boundary data $g_0=g_1=0$. Notice that such u_0 belongs to $H^{(1/2)}$ but not to $H^{(\alpha)}$ with $\alpha>1/2$. This also leads to a piecewise-constant discontinuous weaker solution u.

Table 3 contains results in Example $A^m_{1/2}$ for $\sigma=0.25$. In general they are similar to the results in the previous Table 2; the above mentioned percent is even less: 1.3%.

Table 2. Practical errors and error orders in Example $A_{1/2}$

n	L^2	$SW_h^{1,-1;2}$	L^1
1000	0.041	0.043	0.022
1500	0.324	0.344	0.452
2200	0.328	0.343	0.465
3300	0.327	0.341	0.464
5000	0.329	0.339	0.469

Table 3. Practical errors and error orders in Example $A^m_{1/2}$

n	L^2	$SW_h^{1,-1;2}$	L^1
1000	0.060	0.060	0.021
1500	0.329	0.328	0.462
2200	0.329	0.328	0.465
3300	0.329	0.329	0.464
5000	0.329	0.329	0.467

Example $\mathbf{A_{3/2}}$. We set $u_0(x)=1-2|x|$. Clearly $u_0\in C(\bar{\Omega})$ with the piecewise-constant derivative $D_xu_0(x)=-2\operatorname{sgn} x\in H^{(1/2)}$. In addition, we use $g_0=g_1=0$ without a mismatch of the initial and boundary data. The exact weak solution u is piecewise-linear with the discontinuous piecewise-constant derivatives D_tu and D_xu on $\bar{Q}$.

Table 4 contains results in Example $A_{3/2}$ for $\sigma=0.25$. The final practical orders differ from the theoretical ones within 0.3%, 1.3% and 0.4% for the terms of the error bounds (2.13), (2.14) and (2.15) respectively. We observe that the

L^2, $SW_h^{1,-1;2}$, C_h and $W_h^{0,1;2}$ error orders are non-decreasing while the rest are oscillating. Also the L_h^p and $W_h^{1,0;p}$ errors and orders are clearly better for $p = 1$ than respectively the L^2 and $W_h^{1,0;2}$ ones.

Table 4. Practical errors and error orders in Example $A_{3/2}$

n	L^2	$SW_h^{1,-1;2}$	C_h	$W_h^{1,0;2}$	$W_h^{0,1;2}$	L_h^1	$W_h^{1,0;1}$
1000	1.050E-4	1.043E-4	9.435E-4	0.061	0.060	3.57E-5	0.031
1500	0.998	0.994	0.660	0.330	0.325	1.298	0.474
2200	0.998	0.995	0.662	0.332	0.327	1.297	0.480
3300	0.999	0.996	0.663	0.331	0.327	1.295	0.474
5000	0.999	0.997	0.664	0.332	0.329	1.307	0.471

Note that the final practical orders for the FDM in comparison with the theoretical ones are respectively 0.999 (the same), 1.000, 0.665, 0.333 (all closer), 0.326 (farther), 1.288 and 0.476.

For the FEM and the mesh initial function $v_0 = v_{\sigma_0}^0$, see equation (2.9), with $v_{\sigma_0 i}^0 = 0$, $i = 0, n$, for $\sigma_0 = 1/6$ the results are the same since by chance $v_0 = \widehat{u}_0$ satisfies this equation. For $\sigma_0 = 1/4$, there is no significant difference in results: in particular, the final practical orders are sequentially 0.996, 0.994, 0.666, 0.326, 0.323, 1.297 and 0.468.

Example $\mathbf{A}_{\boldsymbol{\alpha}}$**,** $\boldsymbol{\alpha} = \mathbf{5/2, 7/2}$. We set $u_0(x) = (\operatorname{sgn} x)(2x)^\beta \in C^{\beta-1}(\bar{\Omega})$ with the integer $\beta = \alpha - 1/2$. It has the piecewise-constant higher-order derivative $D_x^\beta u_0(x) = 2^\beta \beta! \operatorname{sgn} x \in H^{(1/2)}$. In addition, we use $g_1(t) = -g_0(t) = 1 + (2t)^2$ for $\alpha = 5/2$ and $g_1(t) = g_0(t) = 1 + 3(2t)^2$ for $\alpha = 7/2$ to ensure matching of the initial and boundary data. The exact solution $u \in C^1(\bar{Q})$ is strong with the piecewise-constant discontinuous 2nd order derivatives $D_x^2 u$, $D_t^2 u$ and $D_x D_t u$ for $\alpha = 5/2$, whereas $u \in C^2(\bar{Q})$ is classical with the piecewise-linear 2nd order derivatives $D_x^2 u$, $D_x^2 u$ and $D_x D_t u$ and the discontinuous 3rd order derivatives (thus u is not too smooth) for $\alpha = 7/2$.

Table 5 contains results in Example $A_{5/2}$. The final practical orders differ from the theoretical ones within 0.16%, 0.6% and 0.35% for the terms of the error bounds (2.13), (2.14) and (2.15) respectively. We see that the L_h^2, $SW_h^{1,-1;2}$, $W_h^{1,0;2}$ and $W_h^{0,1;2}$ error orders are non-decreasing while the rest are oscillating. Once again the L_h^p and $W_h^{1,0;p}$ errors and orders are better for $p = 1$ than $p = 2$.

We take $v_0 = v_{\sigma_0}^0$ with $\sigma_0 = 1/6$ and replace u_0^h by u_0 in (2.9) for $\alpha = 7/2$. Table 6 contains results in Example $A_{7/2}$. The final practical orders differ from the known theoretical ones within only 0.05%-0.14% (we ignore validity of the error bound (2.15) for any $3 \le \alpha < 7/2$ only, not $\alpha = 7/2$). Also the $W_h^{1,0;1}$ error and orders are better than the $W_h^{1,0;2}$ ones.

Example $\mathbf{A}_{\boldsymbol{\alpha}}$**,** $\boldsymbol{\alpha} = \mathbf{1, 2, 3}$. We set $u_0(x) = (\operatorname{sgn} x)|2x|^\beta \in C^{[\beta]}(\bar{\Omega})$ with the half-integer $\beta = \alpha - 1/2$. Notice that $u_0 \notin W^{\alpha,2}(\Omega)$ so that, strictly speaking, the above error bounds are not applicable with these α. But nevertheless u_0

Table 5. Practical errors and error orders in Example $A_{5/2}$

n	L_h^2	$SW_h^{1,-1;2}$	C_h	$W_h^{1,0;2}$	$W_h^{0,1;2}$	L_h^1	$W_h^{1,0;1}$
1000	7.84E-6	7.817E-6	6.312E-5	0.002	0.002	1.956E-6	5.953E-4
1500	1.663	1.662	1.320	0.993	0.988	1.977	1.251
2200	1.664	1.663	1.340	0.995	0.991	1.974	1.266
3300	1.665	1.664	1.326	0.996	0.993	1.980	1.258
5000	1.665	1.664	1.338	0.997	0.994	1.985	1.277

Table 6. Practical errors and error orders in Example $A_{7/2}$

n	L_h^2	$SW_h^{1,-1;2}$	C_h	$W_h^{1,0;2}$	$W_h^{0,1;2}$	$W_h^{1,0;1}$
1000	2.859E-6	1.083E-6	3.749E-6	4.695E-5	4.781E-5	1.166E-5
1500	1.999	1.995	1.994	1.662	1.673	1.966
2200	1.999	1.996	2.000	1.663	1.671	1.976
3300	1.999	1.998	1.998	1.664	1.670	1.979
5000	1.998	2.001	1.997	1.665	1.669	1.980

belongs to the fractional order Nikolskii space $H^{\alpha-\varepsilon,2}(\Omega)$ (as well as to the fractional order Sobolev-Slobodetskii space $W^{\alpha-\varepsilon,2}(\Omega)$) for any $0 < \varepsilon < 1/2$ thus the bounds can be applied for $\alpha - \varepsilon$ in the role of α, and we can expect that the practical error orders almost correspond to α. In addition, we use $g_1(t) = -g_0(t) = \big[(1+2t)^\beta + [(1-2t)^\beta\big]/2,\ 0 \le t < 1/2$, to ensure again matching of the initial and boundary data.

Tables 7, 8 and 9 contain results in Examples A_α respectively for $\alpha = 1, 2, 3$. For $\alpha = 1$, the final practical orders differ from the discussed theoretical ones within 0.4%–4.3%.

Table 7. Practical errors and error orders in Example A_1

n	L_h^2	$SW_h^{1,-1;2}$	C_h	L_h^1
1000	0.003	0.003	0.025	0.001
1500	0.658	0.662	0.320	0.867
2200	0.660	0.663	0.336	0.879
3300	0.661	0.664	0.347	0.877
5000	0.663	0.664	0.319	0.886

For $\alpha = 2$, we set $T = 0.3$ and $\sigma = 0.25$. The final practical orders differ from the theoretical ones within 0%–1.3% for the terms of the error bounds (2.13)–(2.14), and they coincide for the error bound (2.15). All the error orders are increasing except the oscillating C_h one.

For $\alpha = 3$, the final practical orders differ from the theoretical ones within 0.08%–2.3%, and all the error orders are non-decreasing except the last one.

For all $\alpha = 1, 2, 3$, the L_h^p and $W_h^{1,0;p}$ errors and orders are clearly better for $p = 1$ than $p = 2$.

Notice also that the errors for $n = 1000$ in the same norms decrease as α grows from 1/2 to 3.

Table 8. Practical errors and error orders in Example A_2

n	L_h^2	$SW_h^{1,-1;2}$	C_h	$W_h^{1,0;2}$	$W_h^{0,1;2}$	L_h^1	$W_h^{1,0;1}$
1000	3.491E-5	3.491E-5	2.81E-4	0.010	0.010	1.074E-5	0.004
1500	1.329	1.329	0.978	0.657	0.650	1.601	0.867
2200	1.330	1.330	1.004	0.659	0.653	1.618	0.868
3300	1.331	1.331	0.998	0.660	0.656	1.619	0.873
5000	1.332	1.332	1.000	0.662	0.658	1.626	0.888

Table 9. Practical errors and error orders in Example A_3

n	L_h^2	$SW_h^{1,-1;2}$	C_h	$W_h^{1,0;2}$	$W_h^{0,1;2}$	$W_h^{1,0;1}$
1000	3.28E-7	7.908E-7	2.539E-6	5.669E-5	5.637E-5	1.485E-5
1500	1.946	1.990	1.636	1.324	1.321	1.618
2200	1.948	1.991	1.646	1.326	1.324	1.612
3300	1.950	1.991	1.659	1.328	1.326	1.633
5000	1.954	1.991	1.668	1.329	1.328	1.639

3.2 Practical error orders depending on the smoothness of $\mathbf{u_1}$

Our second collection of seven Examples B_β, $\beta = -1/2, 1/2, 3/2, 5/2$ and $\beta = 0, 1, 2$, is chosen to analyze practical error orders depending on the $L^2(\Omega)$-smoothness of order β of u_1. Recall that $\alpha = \beta + 1$ in the error bounds (2.13)–(2.15). The collection begins with Example $B_{-1/2}$ where u_1 is the Dirac δ-function. Other functions u_1 are nothing more than the piecewise-power (or piecewise-linear) functions once again. We assume that $u_0 = 0$ and $f = 0$ in all the examples. We mostly take $T = 0.4$ and $\sigma = 0.5$ and replace u_1^h simply by u_1 in equation (2.6) unless the opposite is explicitly stated.

Example $\mathbf{B_{-1/2}}$. We first take $u_1(x) = \delta(x)$ as the Dirac δ-function concentrated at $x = 0$. Since $\delta = D_x H$, the $L^2(\Omega)$-smoothness of δ is negative and equals $-1/2$. We also set $g_0(t) = g_1(t) = 0$. The exact weaker solution u is piecewise-constant and discontinuous on $\bar{Q}$. We use $u_{1i}^h = \delta_{i,n/2}/h$ on ω_h (for even n) in equation (2.6) according to (2.12). Hereafter $\delta_{i,j}$ is the Kronecker delta.

Table 10 contains results in Example $B_{-1/2}$ for $\sigma = 0.25$. The final practical orders differ from the theoretical one within only 0.4% for the error bound (2.13). All the error orders are oscillating. Also the L^1 error and orders are clearly better than the L^2 ones.

Table 10. Practical errors and error orders in Example $B_{-1/2}$

n	L^2	$SW_h^{1,-1;2}$	L^1
1000	0.031	0.031	0.015
1500	0.334	0.330	0.473
2200	0.331	0.332	0.464
3300	0.333	0.331	0.476
5000	0.332	0.332	0.474

Example $\mathbf{B}_{\boldsymbol{\beta}}$**,** $\boldsymbol{\beta}$**=** $\mathbf{1/2, 3/2, 5/2}$. We take $u_1(x) = \operatorname{sgn} x$ for $\beta = 1/2$, $u_1(x) = 1 - 2|x|$ for $\beta = 3/2$, $u_1(x) = (\operatorname{sgn} x)x^2$ for $\beta = 5/2$ similarly to the corresponding Examples $A_{\beta+1}$. In addition, we take $g_1(t) = -g_0(t) = t$ for $\beta = 1/2$ and $g_1(t) = -g_0(t) = t^3/3 + t/4$ for $\beta = 5/2$ to ensure matching of the initial and boundary data.

The exact solution u is weak and piecewise-linear with discontinuous piecewise-constant derivatives $D_t u$ and $D_x u$ on $\bar{Q}$ for $\beta = 1/2$, $u \in C^1(\bar{Q})$ is strong with the piecewise-constant discontinuous 2nd order derivatives $D_x^2 u$, $D_t^2 u$ and $D_x D_t u$ for $\beta = 3/2$ and $u \in C^2(\bar{Q})$ is classical with the piecewise-linear 2nd order derivatives $D_x^2 u$, $D_t^2 u$ and $D_x D_t u$ and discontinuous 3rd order derivatives (thus u is not too smooth) for $\beta = 5/2$. We use $u_{1i}^h = \operatorname{sgn} x_i$ for $i \neq n/2$ and $u_{1n/2}^h = 0$ in equation (2.6) (for even n) according to (2.10) for $\beta = 1/2$. Table 11 contains results in Example $B_{1/2}$ for $T = 0.2$. The final practical orders differ from the theoretical ones within 0.6%, 5.5% and 0.05% for the terms of the error bounds (2.13), (2.14) and (2.15) respectively. The error orders are non-decreasing except the oscillating last one.

Table 11. Practical errors and error orders in Example $B_{1/2}$

n	L^2	$SW_h^{1,-1;2}$	C_h	$W_h^{1,0;2}$	$W_h^{0,1;2}$	L_h^1	$W_h^{1,0;1}$
1000	1.455E-4	1.464E-4	0.001	0.065	0.063	4.343E-5	0.024
1500	0.986	0.989	0.667	0.312	0.305	1.225	0.444
2200	0.989	0.992	0.667	0.315	0.309	1.247	0.451
3300	0.992	0.994	0.667	0.318	0.312	1.255	0.457
5000	0.994	0.995	0.667	0.320	0.315	1.261	0.456

The same error results are valid also for $u_1(x) = \operatorname{sgn} x - 2x$ together with $g_0(t) = g_1(t) = 0$ since the function $2xt$ solves both the IBVP (2.1)–(2.3) and the FEM (2.5), (2.6) and (2.8) for $f = u_0 = 0$, $u_1(x) = 2x$, the above $g_1(t) = -g_0(t) = t$ and $v_0 = 0$ on $\bar{\omega}_h$.

Notice that quite similar results are valid even for the simplest $u_1(x) \equiv 1$ and $g_0(t) = g_1(t) = 0$ (cp. Example $A_{1/2}^m$). In particular, the final practical orders are sequentially 0.996, 0.997, 0.667, 0.323, 0.319, 1.265 and 0.466. This is not surprising since then due to the mismatch between the second initial condition $D_t u|_{t=0} = u_1$ and zero boundary data the exact weak solution u remains continuous piecewise-linear with discontinuous piecewise-constant derivatives $D_t u$ and $D_x u$. Table 12 contains results in Example $B_{3/2}$. The final practical orders differ from the theoretical ones within only 0.1%, 0.4% and 0.2% for the terms of the error bounds (2.13), (2.14) and (2.15) respectively.

Table 12. Practical errors and error orders in Example $B_{3/2}$

n	L_h^2	$SW_h^{1,-1;2}$	C_h	$W_h^{1,0;2}$	$W_h^{0,1;2}$	L_h^1	$W_h^{1,0;1}$
1000	1.958E-6	1.964E-6	1.569E-5	4.193E-4	4.150E-4	5.745E-7	1.571E-4
1500	1.663	1.664	1.316	0.997	0.991	1.964	1.263
2200	1.664	1.665	1.337	0.998	0.993	1.962	1.277
3300	1.664	1.665	1.325	0.998	0.995	1.970	1.267
5000	1.665	1.666	1.336	0.999	0.996	1.973	1.286

Table 13. Practical errors and error orders in Example $B_{5/2}$

n	L_h^2	$SW_h^{1,-1;2}$	C_h	$W_h^{1,0;2}$	$W_h^{0,1;2}$	$W_h^{1,0;1}$
1000	3.933E-8	5.019E-8	9.829E-8	1.528E-6	1.551E-6	5.019E-7
1500	1.996	1.998	1.982	1.658	1.666	1.944
2200	1.998	1.998	1.989	1.660	1.666	1.954
3300	1.999	1.998	1.990	1.662	1.667	1.959
5000	2.001	1.997	1.995	1.663	1.667	1.965

Table 13 contains results in Example $B_{5/2}$ for $\sigma = 0.25$. The final practical orders differ from the theoretical ones within only 0.15% and 0.25% for the terms of the error bounds (2.13) and (2.14)–(2.15) respectively (we ignore validity of the error bound (2.15) for any $2 \le \beta = \alpha - 1 < 5/2$ only, not $\beta = 5/2$). All the error orders are monotone.

Also the L_h^1 (for $\beta = 1/2, 3/2$) and $W_h^{1,0;1}$ (for $\beta = 1/2, 3/2, 5/2$) errors and orders are better than respectively the L^2 (or L_h^2) and $W_h^{1,0;2}$ ones.

Example $\mathbf{B}_{\boldsymbol{\beta}}$**,** $\boldsymbol{\beta}= \mathbf{0, 1, 2}$. We take $u_1(x) = (\operatorname{sgn} x)|x|^{\beta-1/2}$. Notice that $u_1(x) = (\operatorname{sgn} x)|x|^{-1/2} \notin L^2(\Omega)$ for $\beta = 0$ so that, strictly speaking, the above error bounds are not applicable for $\alpha = 1$. But nevertheless $u_1(x) = 2D_x\sqrt{|x|}$ with $\sqrt{|x|} \in H^{(1-\varepsilon)}$ for any $0 < \varepsilon < 1$, thus the bounds are applicable for $\alpha = 1 - \varepsilon$, and we can expect that the practical error orders almost correspond to $\alpha = 1$. The same functions as $u_1(x) = (\operatorname{sgn} x)|x|^{\beta-1/2}$ for $\beta = 1, 2$ (up to constant multipliers) have already been discussed in Example A_β. In addition, we take

$$g_1(t) = -g_0(t) = \frac{1}{2^{\beta+3/2}(\beta+1/2)}\left[(1+2t)^{\beta+1/2} - (1-2t)^{\beta+1/2}\right], \ 0 \le t < \frac{1}{2}.$$

The above formulas in Examples $B_{1/2}$ and $B_{5/2}$ are particular cases of this one.

We use $u_{1i}^h = u_1(x_i)$ for $i \ne n/2$ and $u_{1n/2}^h = 0$ in equation (2.6) (for even

n) in the case $\beta = 0$. Table 14 contains results in Example B_0. The final practical orders differ from the theoretical ones within only 0.5%–1.15%.

Table 14. Practical errors and error orders in Example B_0

n	L_h^2	$SW_h^{1,-1;2}$	C_h	L_h^1
1000	0.005	0.005	0.056	0.002
1500	0.680	0.652	0.336	0.867
2200	0.678	0.655	0.335	0.869
3300	0.676	0.657	0.335	0.880
5000	0.674	0.659	0.335	0.878

Table 15 contains results in Example B_1 for $\sigma = 0.25$. The final practical orders differ from the theoretical ones within only 0.1%–0.6%.

Table 15. Practical errors and error orders in Example B_1

n	L_h^2	$SW_h^{1,-1;2}$	C_h	$W_h^{1,0;2}$	$W_h^{0,1;2}$	L_h^1	$W_h^{1,0;1}$
1000	3.315E-6	3.267E-6	2.844E-5	0.001	0.001	9.222E-7	5.344E-4
1500	1.338	1.330	1.003	0.664	0.660	1.627	0.882
2200	1.337	1.330	1.014	0.665	0.662	1.651	0.891
3300	1.336	1.331	1.009	0.666	0.663	1.627	0.887
5000	1.335	1.332	1.006	0.666	0.664	1.643	0.888

Table 16 contains results in Example B_2 for $\sigma = 0.25$. The final practical orders differ from the theoretical ones within 1.8%, 0.1% and 2.84% for the terms of the error bounds (2.13), (2.14) and (2.15) respectively. Note that for $\sigma = 0.5$ the percents are respectively 3.35%, 0.33% (both larger) and 0.74% (much smaller). Also the L_h^p (for $\beta = 0, 1$) and $W_h^{1,0;p}$ (for $\beta = 1, 2$) errors and orders are better for $p = 1$ than $p = 2$. Notice also that the errors for $n = 1000$ in the same norms decrease as β grows from $-1/2$ to 2.

Table 16. Practical errors and error orders in Example B_2

n	L_h^2	$SW_h^{1,-1;2}$	C_h	$W_h^{1,0;2}$	$W_h^{0,1;2}$	$W_h^{1,0;1}$
1000	3.850E-8	4.113E-8	1.883E-7	4.936E-6	4.908E-6	1.295E-6
1500	1.965	1.969	1.739	1.334	1.331	1.639
2200	1.966	1.970	1.731	1.334	1.331	1.634
3300	1.966	1.971	1.718	1.334	1.332	1.635
5000	1.964	1.970	1.714	1.334	1.332	1.644

3.3 Practical error orders depending on the smoothness of f

The third collection of seven Examples C_{α_1,α_2}, where $\alpha_1 = -1/2, 1/2, 3/2$ and respectively $\alpha_2 \in \{0, 1\}, \{0, 1, 2\}, \{0, 1\}$, is chosen to analyze practical error orders depending on the dominating mixed smoothness of f, namely, $L^2(\Omega)$-smoothness of order α_1 in x and the generalized $L^1(S)$-smoothness of order α_2 in t. Remind that we consider only f with the separated variables for more clearance; also $f(x,t) = 0$ for $t < t_*$ with some $t_* \in S$. Recall that $\alpha = \alpha_1 + \alpha_2 + 1$ in the error bounds (2.13)–(2.15). The corresponding exact solutions u are piecewise-polynomial from constant to cubic.

For $\alpha_2 = 0$, f is the Dirac δ-function in (x,t) for $\alpha_1 = -1/2$ or in t only for $\alpha_1 = 1/2, 3/2$. These cases are of special interest in connection with utilizing the impulse external forces, in particular, as controls. For brevity, in contrast to above two subsections we do not consider the integer values of α_1.

We assume that $u_0 = u_1 = 0$, $g_0 = g_1 = 0$ and $t_* = 0.1$ in all the examples.

Example $\mathbf{C_{-1/2,0}}$. We first take $f(x,t) = \delta(x, t - t_*)$ as the Dirac δ-function concentrated at the point $(0, t_*)$ with $t_* \in S$. The exact weaker solution u is piecewise-constant and discontinuous on $\bar{Q}$.

Notice that in our computations $(0, t_*) = (x_{n/2}, t_{m_0}) \in \omega_h \times \bar{\omega}^\tau$, for even n and some $0 < m_0 < M$; thus $f_i^{h,\tau,m} = \delta_{i,n/2}\delta_{m,m_0}/(h\tau)$ on $\omega_h \times \bar{\omega}^\tau$, see (2.12).

Table 17 contains results in Example $C_{-1/2,0}$ for $T = 0.5$ and $\sigma = 0.25$. The final practical orders differ from the known theoretical one within only 0.4%. All the error orders oscillate. Also the L^1 error and orders are clearly better than the L^2 ones.

Table 17. Practical errors and error orders in Example $C_{-1/2,0}$

n	L^2	$SW_h^{1,-1;2}$	L^1
1000	0.031	0.031	0.015
1500	0.334	0.330	0.473
2200	0.331	0.332	0.464
3300	0.333	0.331	0.476
5000	0.332	0.332	0.474

Example $\mathbf{C_{-1/2,1}}$. We take $f(x,t) = \delta(x)H(t - t_*)$ as the Dirac δ-function concentrated at the point $(0, t_*)$ with $t_* \in S$. The exact weak solution u is piecewise-linear with discontinuous piecewise-constant derivatives $D_t u$ and $D_x u$ on $\bar{Q}$.

The averages of the cofactors of f are calculated as stated above; in particular, for $z(t) = H(t - t_*)$ with $t_* = t_{m_0}$, we have $z^{\tau,m} = H(t_m - t_*)$ for $m \neq m_0$ and $z^{\tau,m_0} = 1/2$ according to (2.11).

Table 18 contains results in Example $C_{-1/2,1}$ for $T = 0.5$ and $\sigma = 0.5$. The final practical orders differ from the possible theoretical ones within only 0.1%, 2.2% and 0.25% for the terms of the error bounds (2.13), (2.14) and (2.15) respectively. Here "possible" means that the values $(\alpha_1, \alpha_2) = (-1/2, 1)$ were covered in (2.14) only but not in (2.13) and (2.15) though some related extensions were discussed in [10]. The 1st-3rd and 5th error orders are non-decreasing whereas other oscillate. Also the L_h^1 and $W_h^{1,0;1}$ errors and orders are better than respectively the L^2 and $W_h^{1,0;2}$ ones.

Table 18. Practical errors and error orders in Example $C_{-1/2,1}$

n	L^2	$SW_h^{1,-1;2}$	C_h	$W_h^{1,0;2}$	$W_h^{0,1;2}$	L_h^1	$W_h^{1,0;1}$
1000	1.048E-4	1.054E-4	7.517E-4	0.039	0.037	3.992E-5	0.022
1500	0.997	1.000	0.663	0.330	0.322	1.269	0.461
2200	0.997	1.000	0.664	0.334	0.322	1.268	0.471
3300	0.998	1.000	0.664	0.332	0.324	1.283	0.467
5000	0.999	1.000	0.665	0.332	0.326	1.279	0.471

Example $\mathbf{C_{1/2,0}}$. We take $f(x,t) = H(x)\delta(t - t_*)$ with $t_* \in S$. The exact solution u is almost weak and piecewise-linear with discontinuous piecewise-constant derivatives $D_t u$ and $D_x u$ on $\bar{Q}$. Here "the almost weak" solution from the energy class means that $D_t u$ belongs to $L^\infty(S, L^2(\Omega))$ only but not to $C(\bar{S}, L^2(\Omega))$ as $D_x u$, see details in [10]; fortunately this fact does not reduce the error orders.

The averages of the cofactors of f are calculated as stated above. Table 19 contains results in Example $C_{1/2,0}$ for $T = 0.3$ and $\sigma = 0.25$. The final practical orders differ from the theoretical ones within 0.6%, 5.5% and 0.05% for the terms of the error bounds (2.13), (2.14) and (2.15) respectively. All the orders are increasing except the constant C_h one and the oscillating last one. Also the L_h^1 and $W_h^{1,0;1}$ errors and orders are better than respectively the L^2 and $W_h^{1,0;2}$ ones. The results in this and the previous example $C_{-1/2,1}$ are similar that is seemed rather natural due to the form of the error bounds (2.13)–(2.15).

Table 19. Practical errors and error orders in Example $C_{1/2,0}$

n	L^2	$SW_h^{1,-1;2}$	C_h	$W_h^{1,0;2}$	$W_h^{0,1;2}$	L_h^1	$W_h^{1,0;1}$
1000	1.26E-4	1.268E-4	0.001	0.056	0.055	4.343E-5	0.024
1500	0.986	0.989	0.667	0.312	0.305	1.225	0.444
2200	0.989	0.992	0.667	0.315	0.309	1.247	0.451
3300	0.992	0.994	0.667	0.318	0.312	1.255	0.457
5000	0.994	0.995	0.667	0.320	0.315	1.261	0.456

Example $\mathbf{C_{1/2,1}}$. We take $f(x,t) = H(x)H(t - t_*)$ with $t_* \in S$. The exact solution $u \in C^1(\bar{Q})$ is piecewise-quadratic and strong with the piecewise-constant discontinuous 2nd order derivatives $D_t^2 u$, $D_x^2 u$ and $D_x D_t u$ on $\bar{Q}$.

Table 20 contains results in Example $C_{1/2,1}$ for $T = 0.3$ and $\sigma = 0.25$. The final practical orders differ from the theoretical ones within only 0.25%, 1% and 0.35% for the terms of the error bounds (2.13), (2.14) and (2.15) respectively. For $\sigma = 0.5$, the percents are respectively 0.22%, 0.9% and 0.58%. Also the L_h^p and $W_h^{1,0;p}$ errors and orders are better for $p = 1$ than $p = 2$.

Table 20. Practical errors and error orders in Example $C_{1/2,1}$

n	L_h^2	$SW_h^{1,-1;2}$	C_h	$W_h^{1,0;2}$	$W_h^{0,1;2}$	L_h^1	$W_h^{1,0;1}$
1000	1.500E-7	1.518E-7	1.959E-6	6.272E-5	6.214E-5	3.052E-8	1.753E-5
1500	1.663	1.670	1.361	0.983	0.978	1.978	1.254
2200	1.664	1.670	1.289	0.987	0.983	1.971	1.218
3300	1.665	1.669	1.317	0.990	0.987	1.981	1.259
5000	1.665	1.668	1.338	0.992	0.990	1.984	1.257

Example $\mathbf{C_{1/2,2}}$. We take $f(x,t) = H(x)L(t-t_*)$ with $t_* \in S$. Hereafter L is a piecewise-linear function: $L(t) = 0$ for $t \le 0$ and $L(t) = t$ for $t > 0$. Clearly $D_tL(t) = H(t)$. The exact solution $u \in C^1(\bar{Q})$ is piecewise-cubic and strong with the piecewise-linear 2nd order discontinuous derivative $D_x^2 u$ but continuous derivatives $D_t^2 u$ and D_xD_tu on $\bar{Q}$; thus there exist also the piecewise-constant 3rd order mixed derivatives $D_x^2D_tu$ and $D_xD_t^2u$ (that is essential for getting higher order error bounds according to [10]) together with D_t^3u on Q.

Note that, for $t_* = t_{m_0}$, we have $L^{\tau,m} = L(t_m)$ for $m \ne m_0$ and $L^{\tau,m_0} = \tau/6$ according to (2.11). Table 21 contains results in Example $C_{1/2,2}$ for $T = 0.3$ and $\sigma = 0.5$. The final practical orders differ from the theoretical ones within only 0.5% (we ignore validity of the error bound (2.15) for any $0 \le \alpha_1 < 1/2$ only, not $\alpha_1 = 1/2$, when $\alpha_2 = 2$). All the orders are non-decreasing. Also the $W_h^{1,0;p}$ error and orders are clearly better for $p = 1$ than $p = 2$.

In this example, the similar results are valid for $t_* = 0$ as well. In particular, for $T = 0.2$ and $\sigma = 0.25$, the final practical orders are sequentially 2.000, 2.000, 2.000, 1.671, 1.663 and 1.962.

Table 21. Practical errors and error orders in Example $C_{1/2,2}$

n	L_h^2	$SW_h^{1,-1;2}$	C_h	$W_h^{1,0;2}$	$W_h^{0,1;2}$	$W_h^{1,0;1}$
1000	1.048E-8	9.055E-9	3.783E-8	4.761E-7	6.723E-7	1.644E-7
1000	1.996	1.995	1.981	1.663	1.658	1.947
1500	1.997	1.996	1.983	1.664	1.660	1.952
2200	1.998	1.997	1.986	1.665	1.662	1.957
3300	1.998	1.998	1.990	1.665	1.663	1.962

Example $\mathbf{C_{3/2,0}}$. We take $f(x,t) = (1-2|x|)\delta(t-t_*)$ with $t_* \in S$. The exact solution u has the same properties as listed above in Example $C_{1/2,1}$.

Note that, for $w(x) = 1-2|x|$ and even n, we have $w_i^h = w(x_i)$ for $i \ne n/2$ and $w_{n/2}^h = 1-2h/3$ according to (2.10). Table 22 contains results in Example $C_{3/2,0}$ for $T = 0.3$ and $\sigma = 0.25$. The final practical orders differ from the theoretical ones within only 0.22%, 0.7% and 0.05% for the terms of the error bound (2.13), (2.14) and (2.15) respectively. For $\sigma = 0.5$, the percents are respectively 0.16%, 1% and 0.7%. Also the L_h^p and $W_h^{1,0;p}$ errors and orders are better for $p = 1$ than $p = 2$. The results in this and the above example $C_{1/2,1}$ are similar in full accordance with the error bounds (2.13)–(2.15).

Table 22. Practical errors and error orders in Example $C_{3/2,0}$

n	L_h^2	$SW_h^{1,-1;2}$	C_h	$W_h^{1,0;2}$	$W_h^{0,1;2}$	L_h^1	$W_h^{1,0;1}$
1000	3.478E-7	3.439E-7	3.880E-6	1.465E-4	1.451E-4	8.521E-8	3.850E-5
1500	1.666	1.659	1.356	0.990	0.985	1.957	1.276
2200	1.666	1.661	1.287	0.992	0.988	1.953	1.238
3300	1.666	1.662	1.312	0.994	0.991	1.970	1.274
5000	1.666	1.663	1.334	0.995	0.993	1.971	1.270

Example $\mathbf{C_{3/2,1}}$. We take $f(x,t) = (1-2|x|)H(t-t_*)$ with $t_* \in S$. Similarly to above Example $C_{1/2,2}$, the exact solution $u \in C^1(\bar{Q})$ is piecewise-cubic and strong with the piecewise-linear 2nd order discontinuous derivative $D_t^2 u$ but continuous derivatives $D_x^2 u$ and $D_x D_t u$ on $\bar{Q}$; thus there exist also the piecewise-constant 3rd order mixed derivatives $D_x^2 D_t u$ and $D_x D_t^2 u$ together with $D_x^3 u$ on Q.

Table 23 contains results in Example $C_{3/2,1}$ for $T = 0.3$ and $\sigma = 0.25$. The final practical orders equal the theoretical ones for the terms of the error bound (2.13) and differ within only 0.22% and 0.1% for the terms of the error bounds (2.14) and (2.15) respectively (we ignore validity of the error bound (2.15) for any $0 \le \alpha_1 < 3/2$ only, not $\alpha_1 = 3/2$, when $\alpha_2 = 1$). As usual, the $W_h^{1,0;1}$ error and orders are better than the $W_h^{1,0;2}$ ones. The results in this and the above example $C_{1/2,2}$ are similar in full accordance with the error bounds (2.13)–(2.15).

Table 23. Practical errors and error orders in Example $C_{3/2,1}$

n	L_h^2	$SW_h^{1,-1;2}$	C_h	$W_h^{1,0;2}$	$W_h^{0,1;2}$	$W_h^{1,0;1}$
1000	1.053E-8	2.652E-8	1.953E-8	3.439E-7	3.500E-7	6.066E-8
1500	1.999	2.000	1.994	1.659	1.669	1.970
2200	1.999	2.000	2.000	1.661	1.669	1.972
3300	1.999	2.000	1.999	1.662	1.668	1.978
5000	2.000	2.000	1.998	1.663	1.668	1.984

Finally, we emphasize that the closeness in general of the above results in Examples A_α, B_β and C_{α_1,α_2} in the cases $\alpha = \beta + 1 = \alpha_1 + \alpha_2 + 1$ for $\alpha = 1/2, 3/2, 5/2, 7/2$ is caused, of course, by the similar regularity properties and the precise forms of the exact solution u.

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