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Regions of Existence and Uniqueness for Singular Two-Point Boundary Value Problems

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Abstract. A monotone iterative technique with lower and upper solutions is presented to identify the regions of existence for the solutions of singular two-point boundary value problems

$$y''(x) + \frac{p'(x)}{p(x)}y'(x) = f(x, y(x)), \quad x \in [0, b],$$
$$y'(0) = 0, \quad Ay(b) + By'(b) = C, \quad A > 0, B \geq 0, C \geq 0,$$

without requiring the monotonicity conditions on $f(x, y)$. Under an additional condition on $f(x, y)$, uniqueness of the solution is also established. These existence and uniqueness results are constructive and complement the existing results. Four examples including some engineering problems are given to illustrate the applicability of the proposed approach.

Keywords: singular boundary value problem, method of lower and upper solutions, existence and uniqueness, monotone iterative technique.

AMS Subject Classification: 34B15; 47H10; 65L10.

1 Introduction

Many axisymmetric problems in science and engineering lead to the nonlinear elliptic partial differential equations subjected to the mixed boundary conditions. For the steady radial solutions, such problems can be reduced to the following two-point boundary value problems (BVPs)

$$\begin{aligned} &y''(x) + \frac{p'(x)}{p(x)}y'(x) = f(x, y(x)), \quad 0 \leq x \leq b, \\ &y'(0) = 0, \quad Ay(b) + By'(b) = C, \quad C \geq 0, \end{aligned} \tag{1.1}$$

where $A > 0$ and $B \geq 0$. Without loss of generality, the condition $C \geq 0$ is imposed in this work since the transformation $y \to -y$ can be applied when $C < 0$. Here, $f(x, y)$ is assumed to be continuous on $\mathbf{Q}$, a subset of the region $[0, b] \times \mathbb{R}$, and may be singular in y. Further, $p(x) = x^m e^{g(x)}$ with $m \geq 0$, where $g(x)$ is real and analytic in $\{x : |x| < r\}$ for some $r > b$. This implies that $p(x) \in C[0, b] \cap C^1(0, b]$, $p(x) > 0$ in $(0, b]$, $p(0) = 0$ for $m > 0$, and $xp'(x)/p(x)$ is nonnegative at $x = 0$. Similar conditions were given in [26, 27, 28]. Hence, BVP (1.1) becomes singular at $x = 0$ if $m > 0$. Under the above conditions on $p(x)$ and $f(x, y)$, the boundary condition $y'(0) = 0$ is appropriate regardless that the integral of $1/p(x)$ from 0 to b is bounded or not, see [11, 33].

For the case where $p(x) = x^m$ and $B = 0$, the existence and uniqueness results for this type of problems have been studied under various assumptions on $f(x, y)$ [12, 14, 15, 17, 29]. Later works discussing the solvability of a slight generalization of BVP (1.1) required $f(x, y)$ to be continuous, negative, and non-decreasing in y for all $y > 0$ [5] or continuous and positive for all $y \in (0, C/A]$ [6]. The methods used therein mainly depend on the approximation theory and various fixed point theorems. For more details, see the recent review article [33] and references therein.

Over the last decades, the method of upper and lower solutions has been proved to be a powerful tool for constructing the lower and upper boundaries of a finite and closed region where the solutions to BVPs exist, see [8, 19, 21] and references therein. By applying this method and fixed-point theorems, existence and/or uniqueness of the solutions to BVP (1.1) or a slight generalization of it have been discussed extensively in [13, 20] for $p(x) = x^m$ and in [1, 2, 3, 16, 22, 23, 24, 25, 30]; see also the survey by Agarwal and his colleagues [4, 33] and references therein. While in [7, 31], this method with monotone iterative technique has been applied to establish such results for $p(x) = x^m$ and $B = 0$. This idea was also used by Pandey and his coworkers [26, 27, 28] to establish the existence-uniqueness results for BVP (1.1). Recently, similar results for such BVPs have been given for $p'(x) = 0$ in [10] and for $p(x) = x^m e^{nx}$ in [9, 32, 34]. In most of these works, the nonlinearity $f(x, y)$ is assumed to be monotonically increasing or decreasing in y in such regions [7, 9, 10, 13, 26, 27, 28, 31, 32, 34]. However, only few provided a systematic approach for constructing the boundaries of regions containing solutions to BVP (1.1) with $p(x) = x^m$ [13] or $p(x) = x^m e^{nx}$ [32].

The purpose of this work is to extend and generalize the constructive approach introduced in [13, 32] to the more general BVP (1.1) by removing the monotonicity conditions on $f(x, y)$. Different from [13, 32], the existence results are proved in the region formed by the lower and upper solutions using the monotone iterative method. The only restrictions are that $f(x, y)$ is continuous in $x \in [0, b]$ and $\partial f/\partial y$ is continuous in this region. Additional sufficient condition on $f(x, y)$ guaranteeing the uniqueness of solution is also established. These existence and uniqueness results also complement the existing works of [26, 27, 28]. Four examples including some real life applications are given to illustrate the theoretical results.

2 Properties of Green's function

Consider the following linear homogeneous BVP

$$L_x[y(x)] \equiv y''(x) + \frac{p'(x)}{p(x)} y'(x) + \lambda y(x) = 0, \quad 0 \le x \le b, \tag{2.1a}$$

$$y'(0) = 0, \quad Ay(b) + By'(b) = C, \tag{2.1b}$$

where λ is a real constant and $p(x)$ satisfies the same conditions given in BVP (1.1). If $u(x;\lambda)$ and $v(x;\lambda)$ are two linearly independent solutions of differential equation (2.1a) and satisfy $u'(0;\lambda) = 0$ and $Av(b;\lambda) + Bv'(b;\lambda) = 0$, then the Green's function $G(x,s)$ for BVP (2.1a)–(2.1b) can be expressed as

$$G(x,s) = \frac{1}{W(s;\lambda)} \begin{cases} u(x;\lambda)v(s;\lambda), & 0 \le x \le s \le b, \\ u(s;\lambda)v(x;\lambda), & 0 \le s \le x \le b, \end{cases} \tag{2.2}$$

where $W(s;\lambda)$ is the Wronskian defined by

$$W(s;\lambda) = v(s;\lambda)u'(s;\lambda) - u(s;\lambda)v'(s;\lambda) = \frac{p(b)}{p(s)} W(b;\lambda),$$

such that $L_x[G(x,s)] = -\delta(x-s)$. Here $\delta(x)$ is the Dirac delta function. Note that $p(s)W(s;\lambda)$ is a non-zero constant for all $s \in [0,b]$ since $u(x;\lambda)$ and $v(x;\lambda)$ are linearly independent. Moreover, from $W(b;\lambda) = v(b;\lambda)u'(b;\lambda) - u(b;\lambda)v'(b;\lambda)$ and $Av(b;\lambda) + Bv'(b;\lambda) = 0$, it follows that

$$v(b;\lambda) = \frac{BW(b;\lambda)}{Au(b;\lambda) + Bu'(b;\lambda)}, \tag{2.3a}$$

$$v'(b;\lambda) = -\frac{AW(b;\lambda)}{Au(b;\lambda) + Bu'(b;\lambda)}. \tag{2.3b}$$

The sign-preserving property of Green's function for boundary value problems plays an important role in monotone iterative method. Pandey and his coworkers [26,27] have proved this property for BVP (2.1a)–(2.1b) using eigenfunction expansion. Here, a totally different approach is used to prove such a property as shown below.

Lemma 1. *If $u(x;\lambda)$ satisfies the differential equation* (2.1a) *with $u'(0;\lambda) = 0$, then $u(x;\lambda) > 0$ and $Au(b;\lambda) + Bu'(b;\lambda) > 0$ for all $x \in [0,b]$ provided $A > 0, B \ge 0$, and $\lambda < \lambda_1$, where λ_1 is positive and the first zero of $Au(b;\lambda) + Bu'(b;\lambda)$.*

Proof. First, we will prove that $u(x;\lambda) > 0$ for all $x \in [0,b]$ if $\lambda < \lambda_1$. As shown in [27], $u(0;\lambda) \ge 0$ for any real λ. This, combined with the fact that $u'(0;\lambda) = 0$ and all nontrivial solutions of $L_x[u(x)] = 0$ have only simple zeros in $[0,b]$ [18, p. 212], gives $u(0;\lambda) > 0$ for any real λ. When $\lambda = 0$, it is easy to show that $u(x;0)$ is constant on $[0,b]$ and then set without loss of generality that $u(x;0) = 1$ since $u(0;\lambda) > 0$. For $\lambda < 0$, it follows from $u(0;\lambda) > 0$, $u'(0;\lambda) = 0$, and equation (2.1a) that $u''(0;\lambda) > 0$. Suppose that $u(c;\lambda) \le 0$ for some point $c \in (0, b]$. As $u(x;\lambda) \in C^1[0,b]$ and $u''(0;\lambda) > 0$, there exists a point $d \in (0,b]$ such that $u(d;\lambda) > 0, u'(d;\lambda) = 0$, and $u''(d;\lambda) < 0$, which violates the differential equation (2.1a) at the point d. Hence, $u(x;\lambda) > 0$ on $[0,b]$ for $\lambda < 0$. For $0 < \lambda < \lambda_1$, $u(x;\lambda)$ does not change sign in $x \in [0,b]$ [26, 27]. This and $u(0;\lambda) > 0$ require $u(x;\lambda) > 0$ on $[0,b]$ if $0 < \lambda < \lambda_1$. Hence, $u(x;\lambda) > 0$ on $[0,b]$ for $\lambda < \lambda_1$.

Next, we show that $Au(b;\lambda) + Bu'(b;\lambda) > 0$ if $\lambda < \lambda_1$. As shown above, $u(x;0) = 1$ on $[0,b]$, which implies that $Au(b;0) + Bu'(b;0) > 0$. Assume $Au(b;\lambda_0) + Bu'(b;\lambda_0) \le 0$ for some $\lambda_0 < \lambda_1$. Since $Au(b;\lambda) + Bu'(b;\lambda)$ is an analytical function of λ, so there exists a constant $\bar{\lambda}$ in $[\lambda_0, 0)$ if $\lambda_0 < 0$ or $(0, \lambda_0]$ if $\lambda_0 > 0$ such that $Au(b;\bar{\lambda}) + Bu'(b;\bar{\lambda}) = 0$. This implies that $\bar{\lambda} < \lambda_1$ is the first zero of $Au(b;\lambda) + Bu'(b;\lambda)$, which is a contradiction. Therefore, $Au(b;\lambda) + Bu'(b;\lambda) > 0$ for $\lambda < \lambda_1$ and this completes the proof. □

Lemma 2. *For all $x, s \in [0,b], A > 0, B \ge 0$, and $\lambda < \lambda_1$, the Green's function $G(x,s)$ is always non-negative, i.e., $G(x,s) \ge 0$.*

Proof. If $W(x;\lambda) > 0$ for all $x \in [0,b]$, then $W(b;\lambda) > 0$. It follows from $B \ge 0$, Equation (2.3a), and Lemma 1 that $v(b;\lambda) \ge 0$ for $\lambda < \lambda_1$. Assume $v(c;\lambda) < 0$ for some $c \in [0,b)$, then there exist a point d in (c,b) such that $v(d;\lambda) = 0$ and $v(x;\lambda) < 0$ for all $x \in [c,d)$. Since $v(x;\lambda) \in C^1[0,b]$, then we have

$$v'(d;\lambda) = \lim_{h\to 0} \frac{v(d;\lambda) - v(d-h;\lambda)}{h} \ge 0.$$

This together with $v(d;\lambda) = 0$ and $u(d;\lambda) > 0$ leads to $W(d;\lambda) = v(d;\lambda)u'(d;\lambda) - u(d;\lambda)v'(d;\lambda) \le 0$, which is impossible since $W(d;\lambda) > 0$. Thus, $v(x;\lambda) \ge 0$ for $\lambda < \lambda_1$. Following the similar analysis, we can show that $v(x;\lambda) \le 0$ if $W(x;\lambda) < 0$ for all $x \in [0,b]$ and $\lambda < \lambda_1$. Therefore, $W(x;\lambda)v(x;\lambda) \ge 0$ and it follows from Equation (2.2) and Lemma 1 that $G(x,s) \ge 0$ for $0 \le x, s \le b$ if $\lambda < \lambda_1$. □

Lemma 3. *For all $x, s \in [0,b], A > 0$, and $B \ge 0$, the Green's function $G(x,s)$ satisfies the following inequality*

$$1 + \lambda \int_0^b G(x,s)\,ds > 0,$$

for $\lambda < \lambda_1$.

Proof. Since $u(x;\lambda)$ and $v(x;\lambda)$ satisfy the differential equation (2.1a), it follows that

$$\lambda p(x)u(x;\lambda) = -\frac{d}{dx}\left(p(x)u'(x;\lambda)\right), \quad \lambda p(x)v(x;\lambda) = -\frac{d}{dx}\left(p(x)v'(x;\lambda)\right),$$

which implies that

$$\frac{\lambda u(x;\lambda)}{W(x;\lambda)} = -\frac{d}{dx}\left(\frac{u'(x;\lambda)}{W(x;\lambda)}\right), \quad \frac{\lambda v(x;\lambda)}{W(x;\lambda)} = -\frac{d}{dx}\left(\frac{v'(x;\lambda)}{W(x;\lambda)}\right).$$

This leads to

$$\lambda \int_0^b G(x,s)\mathrm{d}s = \int_0^x \frac{\lambda v(x;\lambda)u(s;\lambda)}{W(s;\lambda)}\mathrm{d}s + \int_x^b \frac{\lambda u(x;\lambda)v(s;\lambda)}{W(s;\lambda)}\mathrm{d}s$$
$$= -1 - \frac{u(x;\lambda)v'(b;\lambda)}{W(b;\lambda)} = \frac{Au(x;\lambda)}{Au(b;\lambda)+Bu'(b;\lambda)} - 1,$$

using Equation (2.3b). Since $A > 0$, it follows from Lemma 1 that for $\lambda < \lambda_1$

$$1 + \lambda \int_0^b G(x,s)\mathrm{d}s = \frac{Au(x;\lambda)}{Au(b;\lambda)+Bu'(b;\lambda)} > 0.$$

□

3 Region of existence and uniqueness

Since $G(x,s)$ is the Green's function associated with BVP (2.1a)–(2.1b) and $u(x;\lambda)$ and $v(x;\lambda)$ are two linearly independent solutions of differential equation (2.1a) and satisfy $u'(0;\lambda) = 0$ and $Av(b;\lambda)+Bv'(b;\lambda) = 0$, the next lemma follows from the analysis given in [34], Lemma 1, and Lemma 2.

Lemma 4. *The boundary value problem*

$$L_x[y(x)] = h(x), \quad y'(0) = 0, \quad Ay(b) + By'(b) = C,$$

where $0 \le x \le b, A > 0, B \ge 0, C \ge 0$, *and* $h(x) \in C[0,b]$, *has a unique solution given by*

$$y(x) = \frac{Cu(x;\lambda)}{Au(b;\lambda)+Bu'(b;\lambda)} - \int_0^b G(x,s)h(s)ds,$$

provided λ *is none of the zeros of* $Au(b;\lambda) + Bu'(b;\lambda)$. *Furthermore,* $y(x) \ge 0$ *if* $h(x) \le 0$ *and* $\lambda < \lambda_1$.

Let there exist a lower function $v_0(x)$ and an upper solution $u_0(x)$ in $C^2[0,b]$ for BVP (1.1), respectively, such that $v_0 \le u_0$ and satisfy

$$\begin{aligned} &v_0''(x) + \frac{p'(x)}{p(x)}v_0'(x) = F_U(x), \quad 0 \le x \le b, \\ &v_0'(0) = 0, \quad Av_0(b) + Bv_0'(b) = C_L \le C, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &u_0''(x) + \frac{p'(x)}{p(x)}u_0'(x) = F_L(x), \quad 0 \le x \le b, \\ &u_0'(0) = 0, \quad Au_0(b) + Bu_0'(b) = C_U \ge C, \end{aligned} \tag{3.2}$$

where $F_L(x)$ and $F_U(x)$ are given continuous functions on $[0,b]$. Now, define a sequence $\{u_n\}_{n=0}^{\infty}$ generated by

$$\begin{aligned} L_x[u_n(x)] =& \lambda u_{n-1}(x) + f(x, u_{n-1}(x)), \quad 0 \le x \le b, \\ u_n'(0) =& 0, \quad Au_n(b) + Bu_n'(b) = C, \end{aligned} \tag{3.3}$$

for $n \geq 1$ with initial iterate $u_0(x)$. Similarly, using $v_0(x)$ as initial iterate in the following iteration

$$\begin{aligned} L_x[v_n(x)] =& \lambda v_{n-1}(x) + f(x, v_{n-1}(x)), \quad 0 \leq x \leq b, \\ v_n'(0) =& 0, \quad A v_n(b) + B v_n'(b) = C, \end{aligned} \tag{3.4}$$

for $n \geq 1$ leads to another sequence $\{v_n\}_{n=0}^{\infty}$.

In the main theorems that follow, $\mathbf{D} : [0, b] \times [v_0(x), u_0(x)]$ is a closed region formed by the solutions to BVPs (3.1) and (3.2).

Theorem 1. *Suppose that BVP* (1.1) *is well-defined in the region* $\mathbf{Q}$ *and* $f(x, y)$ *satisfies the following condition:*

(H1) *There exist continuous functions* $F_L(x)$ *and* $F_U(x)$ *such that* $F_L(x) \leq f(x, y) \leq F_U(x)$ *for all* $(x, y) \in \mathbf{Q}$.

If $\mathbf{D} \subseteq \mathbf{Q}$, *then every possible solution* $y_p(x)$ *of BVP* (1.1) *that lies entirely in* $\mathbf{Q}$ *must lie entirely in* $\mathbf{D}$.

Proof. Since $(x, y_p) \in \mathbf{Q}$, then (H1) gives $F_L(x) \leq f(x, y_p) \leq F_U(x)$ in $\mathbf{Q}$. This, combined with the fact that $y_p(x)$ and $v_0(x)$ satisfy the BVPs (1.1) and (3.1), respectively, leads to

$$\begin{aligned} &(y_p - v_0)'' + \frac{p'(x)}{p(x)}(y_p - v_0)' = f(x, y_p) - F_U(x) \leq 0, \\ &(y_p - v_0)'(0) = 0, \quad A(y_p - v_0)(b) + B(y_p - v_0)'(b) \geq 0. \end{aligned}$$

Since $\lambda_1 > 0$, it follows from Lemma 4 with $\lambda = 0$ that $y_p(x) \geq v_0(x)$ for all $x \in [0, b]$. Proved in a similar manner is that $y_p(x) \leq u_0(x)$ on $[0, b]$. Hence, every possible solution $y_p(x)$ in $\mathbf{Q}$ satisfying (H1) lies entirely in $\mathbf{D}$. $\square$

Theorem 2 [Existence]. *Suppose* $\partial f / \partial y$ *is continuous in* $\mathbf{Q}$ *and suppose* $f(x, y)$ *satisfies the following conditions:*

(H2) *There exist continuous functions* $F_L(x)$ *and* $F_U(x)$ *such that* $F_L(x) \leq f(x, u_0(x))$ *and* $F_U(x) \geq f(x, v_0(x))$ *for all* $x \in [0, b]$.

(G1) $\lambda + \max\limits_{(x,y) \in \mathbf{D}} \partial f / \partial y \leq 0$.

If $\mathbf{D} \subseteq \mathbf{Q}$, *then two sequences* $\{u_n\}_{n=0}^{\infty}$ *and* $\{v_n\}_{n=0}^{\infty}$ *generated by Equations* (3.3) *and* (3.4) *converge monotonically and uniformly to the solutions of BVP* (1.1) *in* $\mathbf{D}$ *for* $\lambda < \lambda_1$.

Proof. Setting $n = 0$ into Equation (3.3) and subtracting it from Equation (3.2), we get

$$\begin{aligned} &L_x[u_0(x) - u_1(x)] = F_L(x) - f(x, u_0(x)) \leq 0, \\ &(u_0 - u_1)'(0) = 0, \quad A(u_0 - u_1)(b) + B(u_0 - u_1)'(b) \geq 0, \end{aligned}$$

using the condition that $F_L(x) \le f(x, u_0(x))$ for all $x \in [0, b]$ from (H2). Since $\lambda < \lambda_1$, it follows from Lemma 4 that $u_0 \ge u_1$. Now assume that $u_{n-1} \ge u_n$. From Equation (3.3), mean value theorem, (G1), and $u_{n-1} \ge u_n$, we obtain

$$\begin{aligned} L_x[u_n(x) - u_{n+1}(x)] &= f(x, u_{n-1}) - f(x, u_n) + \lambda(u_{n-1} - u_n) \\ &= \left(\lambda + \frac{\partial f}{\partial y}\bigg|_{y=\bar{y}}\right)(u_{n-1} - u_n) \le 0, \\ (u_n - u_{n+1})'(0) = 0, &\quad A(u_n - u_{n+1})(b) + B(u_n - u_{n+1})'(b) = 0, \end{aligned}$$

where $\bar{y}$ is a suitable value between u_{n-1} and u_n. Hence, $u_n \ge u_{n+1}$ from Lemma 4 for $\lambda < \lambda_1$.

Since $u_0 \ge v_0$, assume that $u_n \ge v_0$. From Equations (3.1) and (3.3) with the mean value theorem, conditions (H2) and (G1), it follows that

$$\begin{aligned} &L_x[u_{n+1}(x) - v_0(x)] \le f(x, u_n(x)) - F_U(x) + \lambda(u_n - v_0) \\ &\le f(x, u_n(x)) - f(x, v_0(x)) + \lambda(u_n - v_0) = \left(\lambda + \frac{\partial f}{\partial y}\bigg|_{y=w_0}\right)(u_n - v_0) \le 0, \\ &(u_{n+1} - v_0)'(0) = 0, \quad A(u_{n+1} - v_0)(b) + B(u_{n+1} - v_0)'(b) = 0, \end{aligned}$$

where $w_0 \in [v_0, u_n]$. Then from Lemma 4, $u_{n+1} \ge v_0$ since $\lambda < \lambda_1$ and hence we have

$$u_0 \ge u_1 \ge \cdots \ge u_n \ge u_{n+1} \ge \cdots \ge v_0.$$

By starting with v_0 and using analogous arguments, it is easy to prove

$$v_0 \le v_1 \le \cdots \le v_n \le v_{n+1} \le \cdots \le u_0.$$

Since $u_0 \ge v_0$, assume that $u_n \ge v_n$. The proof of $u_{n+1} \ge v_{n+1}$ follows similar steps and hence

$$u_0 \ge u_1 \ge \cdots \ge u_n \ge u_{n+1} \ge \cdots \ge v_{n+1} \ge v_n \ge \cdots \ge v_1 \ge v_0.$$

This implies that both sequences $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are monotonically non-increasing and non-decreasing, respectively, and are bounded by u_0 and v_0. Therefore by the Dini's theorem, they converge uniformly to $u(x)$ and $v(x)$, respectively, as $n \to \infty$.

The solution to BVP (3.3) can be written as

$$u_n(x) = \frac{Cu(x;\lambda)}{Au(b;\lambda) + Bu'(b;\lambda)} - \int_0^b G(x,s)\left[f(s, u_{n-1}(s)) + \lambda u_{n-1}(s)\right] \mathrm{d}s.$$

Then using Lebesgue dominated convergence theorem and taking limit as $n \to \infty$ in the above equation, we get

$$u(x) = \frac{Cu(x;\lambda)}{Au(b;\lambda) + Bu'(b;\lambda)} - \int_0^b G(x,s)\left[f(s, u(s)) + \lambda u(s)\right] \mathrm{d}s,$$

which is a solution to the following BVP

$$
\begin{aligned}
L_x[y(x)] =& f(x,y(x)) + \lambda y(x), \quad 0 \le x \le b,\\
y'(0) =& 0, \quad Ay(b) + By'(b) = C,
\end{aligned}
$$

where $A > 0, B \ge 0, C \ge 0$, and therefore BVP (1.1) since they are equivalent. Similarly, $v(x)$ is also a solution to BVP (1.1). □

Theorem 3 [Existence]. *Suppose $\partial f/\partial y$ is continuous and non-positive for all $(x,y) \in \mathbf{Q}$. If $\mathbf{D} \subseteq \mathbf{Q}$ and* (H2) *holds, then, BVP* (1.1) *has at least one solution in $\mathbf{D}$ with $\lambda = 0$.*

Proof. Since $\lambda = 0$, $\lambda_1 > 0$, $\mathbf{D} \subseteq \mathbf{Q}$, and $\partial f/\partial y \le 0$ for all $(x,y) \in \mathbf{Q}$, then (G1) holds and $\lambda < \lambda_1$. Hence, all the conditions required by Theorem 2 are satisfied and thereby completing the proof. □

Theorem 4 [Existence]. *Suppose $\partial f/\partial y$ is continuous and non-negative for all $(x,y) \in \mathbf{Q}$. In addition, suppose $f(x,y)$ satisfies* (G1) *and*

(H3) *There exist continuous functions $y_s(x), F_L(x)$ and $F_U(x)$ such that $F_L(x) \le f(x,y_s(x)) \le F_U(x)$ for all $x \in [0,b]$.*

If $(x,y_s) \in \mathbf{D} \subseteq \mathbf{Q}$, then BVP (1.1) *has at least one solution in $\mathbf{D}$ for $\lambda < \lambda_1$.*

Proof. Since $(x,y_s) \in \mathbf{D} \subseteq \mathbf{Q}$ and $\partial f/\partial y \ge 0$ for all $(x,y) \in \mathbf{Q}$, then (H3) give (H2). This and (G1) are the conditions required by Theorem 3 and hence the result follows. □

For proving the uniqueness, we need the following lemma which can be easily proved using the analysis similar to Theorem 6 in [7].

Lemma 5. *If $y(x)$ satisfies*

$$
\begin{aligned}
&y''(x) + \frac{p'(x)}{p(x)} y'(x) + k(x)y(x) \le 0, \quad 0 \le x \le b,\\
&y'(0) = 0, \quad Ay(b) + By'(b) \ge 0,
\end{aligned}
$$

where $A > 0, B \ge 0$ and $p(x)$ satisfies the same conditions given in BVP (1.1)*, then, $y(x) \ge 0$ for all $x \in [0,b]$ provided $k(x) < \lambda_1$.*

Theorem 5 [Uniqueness]. *Suppose that all the conditions of Theorem 2 or 3 hold. Then, BVP* (1.1) *has a unique solution in $\mathbf{D}$ if $f(x,y)$ satisfies the following condition:*

(G2) $\min\limits_{(x,y)\in\mathbf{D}} \partial f/\partial y > -\lambda_1$.

Proof. Let $u(x)$ and $v(x)$ be any two solutions to BVP (1.1) in $\mathbf{D}$, then we get

$$
\begin{aligned}
&(u(x)-v(x))'' + \frac{p'(x)}{p(x)}(u(x)-v(x))' - \left.\frac{\partial f}{\partial y}\right|_{y=\bar{y}} (u(x)-v(x)) = 0,\\
&(u(x)-v(x))'(0) = 0, \quad A(u(x)-v(x))(b) + B(u(x)-v(x))'(b) = 0,
\end{aligned}
$$

by using the mean value theorem. By (G2), we find $-\left.\partial f/\partial y\right|_{y=\bar{y}} < \lambda_1$ since $(x, \bar{y}) \in \mathbf{D}$. From Lemma 5, it follows that $u(x) \geq v(x)$. Similarly, $v(x) \geq u(x)$. Hence $u(x) \equiv v(x)$. $\square$

Since $\lambda_1 > 0$, then (G2) holds if $\partial f/\partial y \geq 0$ for all $(x, y) \in \mathbf{D}$. By Theorem 5, we immediately deduce

Corollary 1. **[Uniqueness]** Suppose that all the conditions of Theorem 2 or 4 are satisfied. Then, BVP (1.1) has a unique solution in $\mathbf{D}$ if $\partial f/\partial y \geq 0$ for all $(x, y) \in \mathbf{D}$.

The key point in applying the above theorems and corollaries is to find the region $\mathbf{Q}$ in which the solution to BVP (1.1) may exist. For the case $p(x) = x^m$ and $\lambda < 0$, Ford and Pennline [13] have presented a theorem which can be applied to find the regions where the solutions to BVP (1.1) cannot exist. Recently, Shivanian [32] extended this theorem to the case $p(x) = x^m e^{nx}$ and $\lambda < 0$. The following theorem shows that their results remain valid for more general BVP (1.1).

Theorem 6. *Let $y(x) \in [y_{min}, y_{max}]$ be a continuous solution to BVP* (1.1) *for $x \in [0, b]$, where y_{min} and y_{max} are its minimum and maximum values, respectively. Suppose that $\partial f/\partial y$ is bounded for $x \in [0, b]$ and $y_C = C/A$. Then, for all $x \in [0, b]$ and $\lambda < \lambda_1$, $y_{max} \leq y_C$ if $f(x, y_{max}) \geq 0$ and $y_{min} \geq y_C$ if $f(x, y_{min}) \leq 0$.*

Proof. Since the solution to BVP (1.1) can be expressed as

$$y(x) = \frac{Cu(x;\lambda)}{Au(b;\lambda) + Bu'(b;\lambda)} - \int_0^b G(x,s)\,[f(s,y(s)) + \lambda y(s)]\,\mathrm{d}s,$$

and y_C also satisfies

$$y_C = \frac{Cu(x;\lambda)}{Au(b;\lambda) + Bu'(b;\lambda)} - \int_0^b G(x,s)\,(\lambda y_C)\,\mathrm{d}s,$$

provided λ is none of the zeros of $Au(b;\lambda) + Bu'(b;\lambda)$, then subtracting each other yields

$$y(x) - y_C = \int_0^b G(x,s)\,[\lambda y_C - \lambda y(s) - f(s,y(s))]\,\mathrm{d}s.$$

If $f(x, y_{\max}) \geq 0$, then following the discussion similar to Theorem 5.4 of [13], we find

$$y_C - y_{\max} \geq \lambda\,(y_{\max} - y_C)\int_0^b G(x,s)\mathrm{d}s.$$

This together with Lemma 3 implies that $y_{\max} \leq y_C$ for $\lambda < \lambda_1$. A similar argument can prove that $y_{\min} \geq y_C$ if $f(x, y_{\min}) \leq 0$ and $\lambda < \lambda_1$. $\square$

4 Applications

In this section, four nonlinear singular two-point BVPs are given to illustrate how to construct the lower and upper boundaries of closed regions where their solutions exist by applying the theorems and corollaries mentioned above.

Example 1. The nonlinear singular two-point boundary value problem

$$y'' + y' + \frac{1}{x}y' = \mathrm{e}^y - 1,$$
$$y'(0) = 0, \quad y(1) + y'(1) = 1,$$

has only one solution in $\mathbf{D} : [0,1] \times [0,1]$.

Obviously, $f(x,y) = \mathrm{e}^y - 1 \geq (\leq) 0$ for $y \geq (\leq) 0$ and all x, and $C/A = 1$. Theorem 6 requires that no solution can have a maximum value in $(1,\infty)$ and minimum value in $(-\infty,0)$. Any solution, therefore, must lie in $\mathbf{Q} : [0,1] \times [0,1]$. Using $F_L(x) = F_U(x) = 0$ with $C_L = 0$ and $C_U = 1$, the solutions to BVPs (3.1) and (3.2) are $v_0(x) = 0$ and $u_0(x) = 1$. This forms a region $\mathbf{D} : [0,1] \times [0,1]$ and implies that the inequalities $F_U(x) \geq f(x, v_0(x))$ and $F_L(x) \leq f(x, u_0(x))$ are satisfied on $[0,1]$. Since $\mathbf{D} = \mathbf{Q}$ and $0 < \partial f/\partial y \leq \mathrm{e}$ in such a region, it follows from Theorem 2 and Corollary 1 that the limit function of the sequence generated by Equation (3.3) or (3.4) with $\lambda \leq -\mathrm{e}$ is the unique positive solution to BVP(E1) in $\mathbf{D}$ and hence the only one.

Example 2. The nonlinear singular two-point boundary value problem

$$y'' + \frac{1}{x}y' = \frac{1}{y},$$
$$y'(0) = 0, \quad y(1) = 1,$$

has a unique positive solution in $\mathbf{D} : [0,1] \times [v_0(x), u_0(x)]$, where

$$v_0(x) = \frac{1+x^2}{2}, \quad u_0(x) = \frac{3+x^2}{4}.$$

For the region above the singularity, i.e., $y > 0$ and all x, $f(x,y) = 1/y$ is positive while below the singularity, it is negative. Since $C/A = 1$, it follows from Theorem 6 that all solutions must lie in $\mathbf{Q} : [0,1] \times (0,1]$. In $\mathbf{Q}$, $f(x,y) \geq 1$, which implies that $F_L(x) = 1$. This with the choice $C_U = 1$ leads to $u_0(x) = (3 + x^2)/4$ as the solution to BVP (3.2). Since $u_0(x) \leq 1$ on $[0,1]$, the condition $F_L(x) \leq f(x, u_0(x))$ is trivially satisfied on $[0,1]$. Let $F_U(x) = 2$ and $C_L = 1$, then, the solution to BVP (3.1) is $v_0(x) = (1 + x^2)/2$. Another condition $F_U(x) \geq f(x, v_0(x))$ is also satisfied and $0 < v_0(x) \leq u_0(x)$ on $[0,1]$. Thus, the region $\mathbf{D} : [0,1] \times [v_0(x), u_0(x)] \subset \mathbf{Q}$ is obtained. Furthermore, in $\mathbf{D}$, $\partial f/\partial y < 0$ and its minimum value is -4, which is larger than $-\lambda_1 = -5.781$ [7]. Therefore, Theorems 3 and 5 prove that a unique solution to BVP(E2) exists in $\mathbf{D}$ and is given by the limit of the sequence (3.3) or (3.4) with $\lambda = 0$.

Example 3. The electric potential distribution in electric double layer

$$y'' + \frac{a}{x}y' = \alpha \sinh(y), \quad \alpha > 0,$$
$$y'(0) = 0, \quad y(1) = C, \quad C > 0,$$

where a is 0 or 1, has a unique positive solution in $\mathbf{D} : [0,1] \times [0,C]$.

Clearly, $f(x,y) \geq (\leq) 0$ for $y \geq (\leq) 0$ and $x \in [0,1]$, and $C/A = C$. By Theorem 6, any solution must lie in $\mathbf{Q} : [0,1] \times [0,C]$. Since $\partial f/\partial y > 0$ everywhere and $f(x,0) = 0$, the condition $F_L(x) \leq f(x,y_s) \leq F_U(x)$ in Theorem 4 holds with $y_s = F_L(x) = F_U(x) = 0$. Taking $C_L = 0$ and $C_U = C$, the resulting solutions to BVPs (3.1) and (3.2) are $v_0(x) = 0$ and $u_0(x) = C$, respectively, and forms a region $\mathbf{D} : [0,1] \times [0,C]$. Obviously, $\mathbf{D}$ coincides with $\mathbf{Q}$ and contains (x, y_s) for $x \in [0,1]$. By Theorem 4 and Corollary 1, BVP(E3) has only one solution given by the limit function of the sequence generated by Equation (3.3) or (3.4) with $\lambda \leq -\alpha \cosh(C)$ and this unique solution exists in $\mathbf{D}$.

Example 4. The reactant concentration in a chemical reactor

$$y'' + \frac{a}{x}y' = \alpha y^n \exp\left(\frac{\gamma\beta(1-y)}{1+\beta(1-y)}\right), \quad n \geq 1, \quad \alpha, \beta, \gamma > 0,$$
$$y'(0) = 0, \quad y(1) + \frac{1}{Nu}y'(1) = 1, \quad Nu > 0,$$

where a is 0, 1, or 2, and Nu is the Nusselt number, has at least one positive solution in $\mathbf{D_1} : [0,1] \times [v_0(x), 1]$, where

$$v_0(x) = 1 + M\frac{x^2 - 1 - \frac{2}{Nu}}{2(a+2)}, \tag{4.1}$$

if $\beta\gamma \geq n$ and $M(2 + Nu) \leq 2(1+a)Nu$.

Ford and Pennline [13] have found that any positive solution to this problem must lie in $\mathbf{Q} : [0,1] \times [0,1]$ and such a solution is unique if $\beta\gamma < n$. If $\beta\gamma \geq n$, then $\partial f(x,y)/\partial y$ is positive for $y \in [0, y_1)$, but negative for $y \in (y_1, 1]$, where

$$y_1 = \frac{2\beta^2 + 2\beta + \frac{\beta\gamma}{n} - \sqrt{\frac{\beta\gamma}{n}(4\beta^2 + 4\beta + \frac{\beta\gamma}{n})}}{2\beta^2},$$

such that $\partial f(x,y)/\partial y = 0$ at $y = y_1$. In $\mathbf{Q}$, $0 \leq f(x,y) \leq M$, where

$$M = \alpha y_1^n \exp\left(\frac{\gamma\beta(1-y_1)}{1+\beta(1-y_1)}\right).$$

Choosing $F_L(x) = 0, F_U(x) = M$, and $C_U = C_L = 1$ in the BVPs (3.1) and (3.2) yield $v_0(x)$ given by Equation (4.1) and $u_0(x) = 1$. Hence, the conditions in Theorems 1 and 2 are trivially satisfied if $v_0(x) \geq 0$. This holds and $\mathbf{D} : [0,1] \times [v_0(x), 1] \subset \mathbf{Q}$ if $M(2 + Nu) \leq 2(1+a)Nu$. Since $\partial f(x,y)/\partial y$ is continuous in $\mathbf{D}$, it has a maximum L. Therefore, for $\beta\gamma \geq n$, BVP(E4) has at least one positive solution given by the limit of the sequence (3.3) or (3.4) with $\lambda + L \leq 0$ in $\mathbf{D} \subset \mathbf{Q}$ if the condition $M(2 + Nu) \leq 2(1+a)Nu$ holds.

5 Conclusions

A constructive and systematic approach for identifying the regions of existence and uniqueness for the solutions of singular nonlinear two-point boundary value problems has been presented. This existence result is proved using the monotone iterative method with lower and upper solutions, and non-negativity of Green's function without monotonicity conditions on $f(x, y)$. The only restrictions are that $f(x, y)$ is continuous in $x \in [0, b]$ and $\partial f/\partial y$ is continuous in these regions. Additional sufficient condition that ensures uniqueness of solution is also established. Theoretical methods are illustrated on four singular nonlinear problems and complement previous known results.

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