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Exponential Stability of Laminated Beam with Constant Delay Feedback

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Abstract. In this article, we consider a system of laminated beams with an internal constant delay term in the transverse displacement. We prove that the dissipation through structural damping at the interface is strong enough to exponentially stabilize the system under suitable assumptions on delay feedback and coefficients of wave propagation speed.

Keywords: laminated beam, interfacial slip, constant delay, exponential decay.

AMS Subject Classification: 35L56; 93D15; 93D23.

1 Introduction

In this work, we are concerned with a model governing vibrations in a structure made up of two layered beams popularly known as “Laminated beam model”, subjected to an internal constant delay term acting on the transverse displacement. Derived by Hansen et al. [15], the laminated beam model describes the vibrations in a structure consisting of two layered identical beams of uniform thickness stuck together by an adhesive (of negligible thickness), in such a way that a slip is permitted while they are continuously in contact with each other. In the absence of interfering forces, the system of the model takes the following form

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x = 0, \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0, \end{cases} \tag{1.1}$$

with $x \in (0, 1)$ and $t \geq 0$. Here $\rho, I_\rho, G, D, \beta$, and, γ are density, mass moment of inertia, shear stiffness, flexural rigidity, adhesive damping parameter and, adhesive stiffness respectively. Similarly, $w = w(x, t)$ denotes the transverse displacement of the beam from its equilibrium position, $\psi = \psi(x, t)$ is the rotation angle, $3s - \psi$ denotes the effective rotation angle and, $s = s(x, t)$ is proportional to the amount of slip along the interface. The first two equations of (1.1) are derived on the assumption of Timoshenko beam theory and, the third equation describes the dynamics of the slip. Moreover, if $s(x, t)$ is identically zero, then the standard Timoshenko system is recovered. Furthermore, if $\beta \neq 0$, then the adhesion at the interface produces a restoration comparable force to counteract the interfacial slip. Otherwise, in absence of adhesive damping (i.e. $\beta = 0$), the third equation of describes the dynamics of slip of coupled laminated beam without structural damping.

Laminated beams have wider applications in engineering as structures are often made out of more than one beam or plate stuck together using the appropriate substance depending on their intended purposes. Among other applications, the closest examples one can think of in recent times are the layered glass gorilla screen protection for smart gadgets, windscreens, among others. Being a controlled system, stability is very important. Thus, many researchers among mathematicians and engineers have focused a lot of attention on the study of well-posedness and more importantly, the stability behavior of this differential model, majorly by exploiting different damping mechanisms introduced to the system. We discus some of the results below.

The asymptotic behavior of system (1.1) with boundary feedback controls of the form

$$\begin{cases} w(0,t) = \psi(0,t) = s(0,t), \quad (\psi - w_x)(1,t) = k_1 w_t(1,t), \\ (3s_x - \psi_x)(1,t) = -k_2(3s_t - \psi_t)(1,t), \end{cases}$$

was studied by Wang et al. [32]. The authors established an exponential stability of the system provided that $r_1 = \sqrt{\frac{\rho}{G}} \neq \sqrt{\frac{I_\rho}{D}} = r_2,\ k_i \neq r_i (i = 1, 2)$. Interestingly, Tatar [31] and Mustafa [24] obtained the result in [32] under weaker conditions on the parameters ρ, G, I_ρ, and D. Some related results were also obtained by Cao et al [10] with different boundary controls.

Apart from stabilization through boundary damping mechanisms, researchers have considered other interesting damping techniques. For example, Raposo [28] introduced extra linear frictional damping terms in the first two equations of (1.1) in addition to structural damping, and proved exponential stability without further restrictions. Later Apalara et al. [9] established that a single linear frictional damping in the effective rotation angle is sufficient for exponential decay in case of equal wave speeds. Similarly, in [2], the authors consider system (1.1) with structural damping, and prove that if it is coupled with boundary feedback controls acting through complementary displacements, then no further dissipation or restrictions on parameters are required for exponential decay, otherwise the assumption of equal wave speeds is necessary.

Regarding dissipation through material damping, for laminated beam with infinity memory, we mention the work in [18], in which with only structural damping and suitable assumptions on the relaxation function, the authors established general exponential decay results in case of equal wave speeds and polynomial stability otherwise. For earlier results concerning stabilization of laminated beam through viscoelastic damping, we refer the reader to [11,21,25]. Furthermore, regarding stabilization through thermal effects, we cite the result in [20]. The authors investigated a thermoelastic laminated beam with past history, and proved that in presence of structural damping, the solution decays exponentially and polynomially without any restrictions on the parameters. For a system without structural damping, exponential and polynomial decay of the solution are possible in case of equal wave speeds, otherwise, the system lacks exponential stability. Other interesting results about damping through thermal effects can be found in [4, 5, 14] for thermoelasticity, and [19] for thermoelasticity of type III. In all these works, authors mainly established that the system decays exponentially in the case of equal wave speeds and polynomially otherwise, with and without structural damping.

In control systems, time delays are inherent since propagation and transport of material and/or information are involved. Time delay may manifest in form of lags between the input and processing the output, or lags in attaining or restoring the desired system stability after perturbations due to internal or external factors, among others. To explicitly analyse the delay effect on physical properties especially stability, it is preferred that control systems are modeled and represented by delay differential equations. Although there are isolated cases where that voluntary inclusion of delay may benefit control (see [1]) or may not significantly disturb the general system stability, for instance, in [23], time lags have been established as one of the underlying causes of instability and deterioration of the system performance. For example, consider the following system of wave equation

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0, & \text{in } \Omega \times (0, \infty), \\ \varphi = 0, & \text{on } \Gamma_0 \times (0, \infty), \\ \dfrac{\partial \varphi}{\partial \nu} = -\mu_1 \varphi_t - \mu_2 \varphi_t(x, t-\tau), & \text{on } \Gamma_1 \times (0, \infty), \end{cases} \tag{1.2}$$

where $\varphi = \varphi(x,t)$, $\Omega \subset \mathbb{R}^2$ is open and bounded having a smooth boundary $\partial\Omega \equiv \Gamma_0 \cup \Gamma_1$ and, $\nu = \nu(x)$ is the unit normal to $\partial\Omega$. It is long established that in the absence of delay ($\mu_2 = 0, \mu_1 > 0$), the system (1.2) is exponentially stable, see [16,17,34] .Whereas, on including delay ($\mu_2 > 0$), Nicaise et al. [26] established that the solution decays exponentially provided that $\mu_2 < \mu_1$, and in case of a reversed scenario ($\mu_2 \geq \mu_1$), the authors proved that the system solutions become chaotic by introducing a correlating sequence of delays to the solution. Similar conclusions were reached by [12,33]. For more works regarding constant time delay effect on stability, we refer the reader to [3, 7, 27, 29] and references therein.

In the dynamic Timoshenko beam model, the amplitude of vibrations of the complementary displacements vanishes due to damping. A constant time delay translates into a forward phase shift increasing early time response, which is seen to cause frequency dispersion in displacements [22]. This may require stronger damping to counteract the longer time needed for decay. This delay effect is inherent in the laminated beam model as it is derived on assumption of Timoshenko beam theory. The presence of structural damping in a laminated beam provides some dissipation, which is sufficient for exponential stability in absence of delay on assumption of equal wave speeds [6, 8]. It is yet to be established if the internal structural damping can still solely stabilize the system in presence of delay, rather authors have chosen other damping mechanisms. For instance, Feng [13], considered a laminated beam with three internal constant delay feedbacks, with help of three external boundary controls and some conditions on the system parameters, he established exponential decay result. Seghour et al. [30] on the other hand, investigated a thermoelastic laminated beam with neutral delay in dynamics of slip equation, and established uniform stability provided $\rho = GI_\rho$. The required dissipation was obtained through thermal effects in addition to linear frictional damping in the transverse displacement.

Considering all the above observations, a natural question arises. Is exponential decay achievable for a classical laminated beam system with constant delay without additional internal damping mechanisms or dissipation through boundary controls? If yes, under what conditions? Our concern is to answer this question affirmatively. Precisely, we consider a system of laminated beam with constant delay term acting on the transverse displacement:

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x + \mu w_t(x, t-\tau) = 0, & \text{in } (0,1) \times (0,\infty), \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, & \text{in } (0,1) \times (0,\infty), \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0, & \text{in } (0,1) \times (0,\infty), \\ w_t(x,-t) = f_0(x,t), & \text{in } (0,1) \times (0,\tau), \\ w(x,0) = w_0,\ s(x,0) = s_0,\ \psi(x,0) = \psi_0, & \text{in } (0,1), \\ w_t(x,0) = w_1,\ s_t(x,0) = s_1,\ \psi_t(x,0) = \psi_1, & \text{in } (0,1), \\ w(0,t) = s_x(0,t) = \psi_x(0,t) = 0, & \text{in } (0,\infty), \\ w_x(1,t) = s(1,t) = \psi(1,t) = 0, & \text{in } (0,\infty), \end{cases} \tag{1.3}$$

where, $w_0, w_1, \psi_0, \psi_1, s_0, s_1, f_0$ is the initial data which belongs to an appropriate space, $\tau > 0$ is time delay and the none zero real number μ is the weight of delay. With some restrictions on μ, we prove that the adhesive damping is strong enough to stabilize the system exponentially, even in presence of delay without any other additional damping or boundary controls, provided the assumption of equal wave propagation speed ($GI_\rho = \rho D$) holds.

The rest of the paper is organized as follows. We present some preliminaries in Section 2. In Section 3, we state and prove some technical lemmas and finally in Section 4, we discuss the stability result.

2 Preliminaries

We proceed by introducing the following new variable as in [26].

$$z(x,\sigma,t) = w_t(x, t-\tau\sigma) \qquad \text{in } (0,1) \times (0,1) \times (0,\infty).$$

It follows directly that z satisfies

$$\tau z_t(x,\sigma,t) + z_\sigma(x,\sigma,t) = 0 \qquad \text{in } (0,1) \times (0,1) \times (0,\infty).$$

Consequently, the system (1.3) is equivalent to

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x + \mu z(x,1,t) = 0, & \text{in } (0,1) \times (0,\infty), \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, & \text{in } (0,1) \times (0,\infty), \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0, & \text{in } (0,1) \times (0,\infty), \\ \tau z_t(x,\sigma,t) + z_\sigma(x,\sigma,t) = 0, & \text{in } (0,1) \times (0,1) \times (0,\infty), \\ z(x,0,t) = w_t(x,t), & \text{in } (0,1) \times (0,\infty), \\ z(x,\sigma,0) = f_0(x,\tau\sigma), & \text{in } (0,1) \times (0,1), \\ w(x,0) = w_0,\ s(x,0) = s_0,\ \psi(x,0) = \psi_0, & \text{in } (0,1), \\ w_t(x,0) = w_1,\ s_t(x,0) = s_1,\ \psi_t(x,0) = \psi_1, & \text{in } (0,1), \\ w(0,t) = s_x(0,t) = \psi_x(0,t) = 0, & \text{in } (0,\infty), \\ w_x(1,t) = s(1,t) = \psi(1,t) = 0, & \text{in } (0,\infty). \end{cases} \tag{2.1}$$

The energy of the solution to the system (2.1) is given by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\int_0^1 \left[\rho w_t^2 + I_\rho(3s_t - \psi_t)^2 + D(3s_x - \psi_x)^2 + 3I_\rho s_t^2 + 3Ds_x^2\right] dx \\ & + \frac{1}{2}\int_0^1 \left[4\gamma s^2 + G(\psi - w_x)^2 + \tau|\mu| \int_0^1 z^2(x,\sigma) d\sigma\right] dx. \end{aligned} \tag{2.2}$$

On the existence, uniqueness, and smoothness of solution of problem (2.1), we introduce the vector function $\Phi = (w, u, \xi, v, s, z)^T$; $u = w_t$, $\xi = 3s - \psi$, $v = \xi_t$, and $y = s_t$, and thereby transform system (2.1) to

$$\begin{cases} \frac{d}{dt}\Phi(t) = \mathcal{A}\Phi(t), & t > 0, \\ \Phi(0) = \Phi_0 = (w_0, w_1, 3s_0 - \psi_0, 3s_1 - \psi_1, s_0, s_1, f_0)^T, \end{cases} \tag{2.3}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\Phi = \begin{pmatrix} u \\ -\frac{1}{\rho}\Big(G\left(3s - \xi - w_x\right)_x + \mu z(x,1)\Big) \\ v \\ \frac{1}{I_\rho}\Big(D\xi_{xx} + G\left(3s - \xi - w_x\right)\Big) \\ y \\ \frac{1}{I_\rho}\Big(Ds_{xx} - G\left(3s - \xi - w_x\right) - \frac{4\gamma}{3}s - \frac{4\beta}{3}y\Big) \\ -\frac{1}{\tau}z_\sigma(x,\sigma) \end{pmatrix}.$$

We consider the following spaces

$$H_a^1 = \{v : v \in H^1(0,1) \ : v(0) = 0\}, \quad H_b^1 = \{v : v \in H^1(0,1) \ : v(1) = 0\}$$

and let

$$\begin{aligned}\mathcal{H} :=& H_a^1(0,1) \times L^2(0,1) \times H_b^1(0,1) \times L^2(0,1) \times H_b^1(0,1) \times L^2(0,1) \\ &\times L^2\left((0,1) \times (0,1)\right)\end{aligned}$$

be the Hilbert space equipped with the following inner product

$$\begin{aligned}(\Phi, \tilde{\Phi})_{\mathcal{H}} =& \rho \int_0^1 u\tilde{u}dx + G \int_0^1 (3s - \xi - w_x)\left(3\tilde{s} - \tilde{\xi} - \tilde{w}_x\right) dx + I_\rho \int_0^1 v\tilde{v}dx \\ &+ 3I_\rho \int_0^1 y\tilde{y}dx + D \int_0^1 \xi_x\tilde{\xi}_x dx + 4\gamma \int_0^1 s\tilde{s}dx + 3D \int_0^1 s_x\tilde{s}_x dx \\ &+ \tau|\mu| \int_0^1 \int_0^1 z(x,\sigma)\tilde{z}(x,\sigma)d\sigma dx.\end{aligned}$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{\begin{matrix}\Phi \in \mathcal{H} \mid w \in H^2(0,1) \cap H_a^1(0,1), \quad \xi, s \in H^2(0,1) \cap H_b^1(0,1), \\ u \in H_a^1(0,1), \quad v, y \in H_b^1(0,1), \quad z, z_\sigma \in L^2\left((0,1) \times (0,1)\right), \\ w_x(1) = \xi_x(0) = s_x(0) = 0\end{matrix}\right\}.$$

We observe that $D(\mathcal{A})$ is independent of time $t > 0$. Furthermore, it is obvious that $D(\mathcal{A})$ is dense in $\mathcal{H}$. We have the following well-posedness result.

Theorem 1. *Let $\Phi_0 \in \mathcal{H}$, then there exists a unique weak solution $\Phi \in C(\mathbb{R}^+, \mathcal{H})$ of problem* (2.3)*. Moreover, if $\Phi_0 \in D(\mathcal{A})$, then $\Phi \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Remark 1. Theorem 1 can be proved using the standard semigroup method as established in [3, 4, 13].

3 Technical lemmas

In this section, we state and prove some technical lemmas which are fundamental in the proof of our stability result. We use multiplier technique to establish stability results for the energy of the solution of system (2.1). This requires constructing a suitable Lyapunov functional equivalent to energy as we elaborate in the subsequent section.

Lemma 1. *If (w, ψ, s, z) is a solution of (2.1), then the energy functional E, defined by (2.2) satisfies*

$$E'(t) \leq -4\beta \int_0^1 s_t^2 dx + |\mu| \int_0^1 w_t^2 dx, \qquad \forall t \geq 0. \tag{3.1}$$

Proof. Multiplying the first three equations in the system (2.1) by $w_t, (3s_t - \psi_t)$ and s_t respectively, then integrating each by parts over $(0,1)$ using the boundary conditions, we end up with

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 &\left[\rho w_t^2 + I_\rho(3s_t-\psi_t)^2 + D(3s_x-\psi_x)^2 + 3I_\rho s_t^2 + 3Ds_x^2 + 4\gamma s^2\right]dx\\ &+\frac{1}{2}\frac{d}{dt}\int_0^1\left[G(\psi-w_x)^2\right]dx = -\mu\int_0^1 z(x,1)w_t dx - 4\beta\int_0^1 s_t^2 dx.\end{aligned} \tag{3.2}$$

Similarly, multiplying $(2.1)_4$ by $|\mu|z$, followed by integrating the product over $(0,1)\times(0,1)$ and then using the substitution $z(x,0,t)=w_t$, we obtain

$$\frac{\tau|\mu|}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\sigma)d\sigma dx = -\frac{|\mu|}{2}\int_0^1 z^2(x,1)dx + \frac{|\mu|}{2}\int_0^1 w_t^2 dx. \tag{3.3}$$

Next, merging (3.2) and (3.3), we note that from (2.2),

$$E'(t) = -\mu\int_0^1 z(x,1)w_t dx - 4\beta\int_0^1 s_t^2 dx - \frac{|\mu|}{2}\int_0^1 z^2(x,1)dx + \frac{|\mu|}{2}\int_0^1 w_t^2 dx. \tag{3.4}$$

We now exploit Young's inequality on the first term of (3.4) to obtain

$$-\mu\int_0^1 z(x,1)w_t dx \le \frac{|\mu|}{2}\int_0^1 z^2(x,1)dx + \frac{|\mu|}{2}\int_0^1 w_t^2 dx. \tag{3.5}$$

Consequently, substituting (3.5) in (3.4) completes the proof of (3.1).
□

Lemma 2. *If (w,ψ,s,z) is a solution of (2.1), then the functional F_1, defined by*

$$F_1(t) := -\rho\int_0^1 ww_t dx + \rho\int_0^1 w_t\int_0^x \psi(y)dydx$$

for any $\varepsilon_1 > 0$, satisfies the estimate

$$\begin{aligned}\frac{d}{dt}F_1(t) \le &-\frac{\rho}{2}\int_0^1 w_t^2 dx + \rho\int_0^1 (3s_t-\psi_t)^2 dx + \varepsilon_1\int_0^1 z^2(x,1)dx\\ &+ 9\rho\int_0^1 s_t^2 dx + \left(\frac{G}{2}+\frac{\mu^2}{4\varepsilon_1}\right)\int_0^1(\psi-w_x)^2 dx.\end{aligned} \tag{3.6}$$

Proof. Differentiating F_1, using the first equation in (2.1), integrating by parts the term containing $(\psi - w_x)_x$ and exploiting the fact that $\psi_t = -(3s_t-\psi_t) + 3s_t$, we deduce that

$$\begin{aligned}\frac{d}{dt}F_1(t) = &-\rho\int_0^1 w_t^2 dx + G\int_0^1(\psi-w_x)^2 dx - \rho\int_0^1 w_t\int_0^x (3s_t-\psi_t)(y)dydx\\ &+ 3\rho\int_0^1 w_t\int_0^x s_t(y)dydx - \mu\int_0^1\left(\int_0^x \psi(y)dy - w\right)z(x,1)dx.\end{aligned} \tag{3.7}$$

By Young's and Poincaré's inequalities, the last three terms of (3.7) give

$$\begin{aligned} -\rho \int_0^1 w_t \int_0^x (3s_t - \psi_t)(y)dydx &\le \rho \int_0^1 \left(\int_0^x (3s_t - \psi_t)dy \right)^2 dx + \frac{\rho}{4} \int_0^1 w_t^2 dx \\ &\le \rho \int_0^1 (3s_t - \psi_t)^2 dx + \frac{\rho}{4} \int_0^1 w_t^2 dx, \end{aligned}$$

$$\begin{aligned} 3\rho \int_0^1 w_t \int_0^x s_t(y)dy &\le 9\rho \int_0^1 \left(\int_0^x s_t(y)dy \right)^2 dx + \frac{\rho}{4} \int_0^1 w_t^2 dx \\ &\le 9\rho \int_0^1 s_t^2 dx + \frac{\rho}{4} \int_0^1 w_t^2 dx \end{aligned}$$

and,

$$\begin{aligned} -\mu \int_0^1 \left(\int_0^x \psi(y)dy - w \right) z(x,1)dx &\le \frac{\mu^2}{4\varepsilon_1} \int_0^1 \left(\int_0^x \psi(y)dy - w \right)^2 dx \\ + \varepsilon_1 \int_0^1 z^2(x,1)dx &\le \frac{\mu^2}{4\varepsilon_1} \int_0^1 (\psi - w_x)^2 dx + \varepsilon_1 \int_0^1 z^2(x,1)dx. \end{aligned} \tag{3.8}$$

The combination of (3.7)−(3.8) leads to (3.6). □

Lemma 3. *If* (w, ψ, s, z) *is a solution of (2.1), then the functional* F_2*, defined by*

$$F_2(t) := -I_\rho \int_0^1 (3s_t - \psi_t)(3s - \psi)dx$$

satisfies the estimate

$$\begin{aligned} \frac{d}{dt} F_2(t) \le &- I_\rho \int_0^1 (3s_t - \psi_t)^2 dx + \frac{3D}{2} \int_0^1 (3s_x - \psi_x)^2 dx \\ &+ \frac{G^2}{2D} \int_0^1 (\psi - w_x)^2 dx. \end{aligned} \tag{3.9}$$

Proof. By direct computations using the second equation in (2.1), we obtain

$$\begin{aligned} \frac{d}{dt} F_2(t) = &- I_\rho \int_0^1 (3s_t - \psi_t)^2 dx + D \int_0^1 (3s_x - \psi_x)^2 dx \\ &- G \int_0^1 (\psi - w_x)(3s - \psi)dx. \end{aligned} \tag{3.10}$$

Exploiting Young's and Poincaré's inequalities, we estimate the last term of (3.10) as follows:

$$\begin{aligned} -G \int_0^1 (\psi - w_x)(3s - \psi)dx &\le \frac{G^2}{2D} \int_0^1 (\psi - w_x)^2 dx + \frac{D}{2} \int_0^1 (3s - \psi)^2 dx \\ &\le \frac{G^2}{2D} \int_0^1 (\psi - w_x)^2 dx + \frac{D}{2} \int_0^1 (3s_x - \psi_x)^2 dx. \end{aligned} \tag{3.11}$$

Consequently, the relation (3.9) follows directly by substituting (3.11) into (3.10). □

Lemma 4. *If (w, ψ, s, z) is a solution of (2.1), then the functional F_3, defined by*

$$F_3(t) := 3I_\rho \int_0^1 s_t s dx + 2\beta \int_0^1 s^2 dx + 3\rho \int_0^1 w_t \int_0^x s(y) dy dx$$

for any $\varepsilon_2 > 0$, satisfies the estimate:

$$\frac{d}{dt}F_3(t) \leq -3D \int_0^1 s_x^2 dx - 3\gamma \int_0^1 s^2 dx + \varepsilon_2 \int_0^1 w_t^2 dx + \frac{9\mu^2}{4\gamma} \int_0^1 z^2(x,1) dx + \left(3I_\rho + \frac{9\rho^2}{4\varepsilon_2}\right) \int_0^1 s_t^2 dx. \tag{3.12}$$

Proof. Differentiating F_3, using (2.1), and then integrating by part the terms containing $(\psi - w_x)_x$ and s_{xx}, we arrive at

$$\begin{aligned} \frac{d}{dt}F_3(t) = &-3D \int_0^1 s_x^2 dx - 4\gamma \int_0^1 s^2 dx + 3\rho \int_0^1 w_t \int_0^x s_t(y) dy dx \\ &+ 3I_\rho \int_0^1 s_t^2 dx - 3\mu \int_0^1 z(x,1) \int_0^x s(y) dy dx. \end{aligned} \tag{3.13}$$

Using Young's, Poincaré's and Cauchy-Schwarz inequalities, we have

$$\begin{aligned} -3\mu \int_0^1 z(x,1) \int_0^x s(y) dy dx &\leq \frac{9\mu^2}{4\gamma} \int_0^1 z^2(x,1) dx + \gamma \int_0^1 \left(\int_0^x s(y) dy\right)^2 dx \\ &\leq \frac{9\mu^2}{4\gamma} \int_0^1 z^2(x,1) dx + \gamma \int_0^1 s^2 dx, \end{aligned} \tag{3.14}$$

$$\begin{aligned} 3\rho \int_0^1 w_t \int_0^x s_t(y) dy dx &\leq \varepsilon_2 \int_0^1 w_t^2 dx + \frac{9\rho^2}{4\varepsilon_2} \int_0^1 \left(\int_0^x s_t(y) dy\right)^2 dx \\ &\leq \varepsilon_2 \int_0^1 w_t^2 dx + \frac{9\rho^2}{4\varepsilon_2} \int_0^1 s_t^2 dx, \end{aligned} \tag{3.15}$$

for any $\varepsilon_2 > 0$. Estimate (3.12) follows directly by virtue of (3.13)−(3.15). □

The assumption of equal wave speeds $GI_\rho = \rho D$ plays an important role in the next two lemmas.

Lemma 5. *If (w, ψ, s, z) is a solution of (2.1), then the functional F_4, defined by*

$$F_4(t) := -\int_0^1 (3s_t - \psi_t) w_x dx - \int_0^1 (3s_x - \psi_x) w_t dx + 3\int_0^1 (3s_t - \psi_t) s dx$$

for any $\varepsilon_3 > 0$, satisfies the estimate

$$\begin{aligned} \frac{d}{dt}F_4(t) \leq &-\frac{D}{2I_\rho} \int_0^1 (3s_x - \psi_x)^2 dx + \varepsilon_3 \int_0^1 (3s_t - \psi_t)^2 dx + \frac{9}{\varepsilon_3} \int_0^1 s_t^2 dx \\ &+ \frac{I_\rho \mu^2}{D\rho^2} \int_0^1 z^2(x,1) dx + \left(\frac{G}{I_\rho} + \frac{G^2}{DI_\rho}\right) \int_0^1 (\psi - w_x)^2 dx. \end{aligned} \tag{3.16}$$

Proof. As in the previous Lemmas, direction computations using (2.1), integration by parts the term containing $3s_{xx} - \psi_{xx}$, and exploiting the fact that $w_x = -(\psi - w_x) - (3s - \psi) + 3s$, we end up with

$$\begin{aligned} \frac{d}{dt}F_4(t) = &-\frac{D}{I_\rho}\int_0^1 (3s_x - \psi_x)^2 dx + \frac{G}{I_\rho}\int_0^1 (\psi - w_x)^2 dx + 3\int_0^1 (3s_t - \psi_t)s_t dx \\ &+ \frac{G}{I_\rho}\int_0^1 (\psi - w_x)(3s - \psi)dx + \frac{\mu}{\rho}\int_0^1 (3s_x - \psi_x)z(x,1)dx. \end{aligned} \tag{3.17}$$

Next, Young's and Poincaré's inequalities guarantee the relations

$$3\int_0^1 (3s_t - \psi_t)s_t dx \le \varepsilon_3 \int_0^1 (3s_t - \psi_t)^2 dx + \frac{9}{\varepsilon_3}\int_0^1 s_t^2 dx, \tag{3.18}$$

$$\begin{aligned} \frac{G}{I_\rho}\int_0^1 (\psi - w_x)(3s - \psi)dx &\le \frac{G^2}{DI_\rho}\int_0^1 (\psi - w_x)^2 dx + \frac{D}{4I_\rho}\int_0^1 (3s - \psi)^2 dx \\ &\le \frac{G^2}{DI_\rho}\int_0^1 (\psi - w_x)^2 dx + \frac{D}{4I_\rho}\int_0^1 (3s_x - \psi_x)^2 dx, \end{aligned}$$

and,

$$\frac{\mu}{\rho}\int_0^1 (3s_x - \psi_x)z(x,1)dx \le \frac{D}{4I_\rho}\int_0^1 (3s_x - \psi_x)^2 dx + \frac{I_\rho \mu^2}{D\rho^2}\int_0^1 z^2(x,1)dx, \tag{3.19}$$

for any $\varepsilon_3 > 0$. Estimate (3.16) follows directly by substituting (3.18)–(3.19) into (3.17). □

Lemma 6. *If (w, ψ, s, z) is a solution of (2.1), then the functional F_5, defined by*

$$F_5(t) := \int_0^1 (\psi - w_x)s_t dx - \int_0^1 w_t s_x dx$$

for any $\varepsilon_4, \varepsilon_5 > 0$, satisfies the estimate

$$\begin{aligned} \frac{d}{dt}F_5(t) \le &-\frac{G}{2I_\rho}\int_0^1 (\psi - w_x)^2 dx + \varepsilon_4 \int_0^1 (3s_t - \psi_t)^2 dx + \varepsilon_5 \int_0^1 z^2(x,1)dx \\ &+ \left(\frac{16\gamma^2}{9I_\rho} + \frac{\mu^2}{4\rho^2\varepsilon_5}\right)\int_0^1 s_x^2 dx + \left(3 + \frac{1}{4\varepsilon_4} + \frac{16\beta^2}{9I_\rho}\right)\int_0^1 s_t^2 dx. \end{aligned} \tag{3.20}$$

Proof. Differentiating F_5 then integrating by parts over $(0,1)$ the term containing s_{xt} and, using the substitution $w_{xt} = -(3s_t - \psi_t) - (\psi - w_x)_t + 3s_t$, we arrive at

$$\frac{d}{dt}F_5(t) = \int_0^1 (\psi - w_x)s_{tt} dx - \int_0^1 w_{tt}s_x dx - \int_0^1 (3s_t - \psi_t)s_t dx + 3\int_0^1 s_t^2 dx. \tag{3.21}$$

Next, using (3.21), (2.1) and ingratiating by parts the term containing s_x, we arrive at

$$\frac{d}{dt}F_5(t) = -\frac{G}{I_\rho}\int_0^1 (\psi - w_x)^2 dx + 3\int_0^1 s_t^2 dx - \frac{4\gamma}{3I_\rho}\int_0^1 s(\psi - w_x)dx$$
$$-\frac{4\beta}{3I_\rho}\int_0^1 s_t(\psi - w_x)dx - \int_0^1 (3s_t - \psi_t)s_t dx + \frac{\mu}{\rho}\int_0^1 z(x,1)s_x dx. \quad (3.22)$$

Exploiting Youngs and Poincaré's inequalities, the last four terms of (3.22) are estimated as follows

$$-\frac{4\gamma}{3I_\rho}\int_0^1 s(\psi - w_x)dx \le \frac{G}{4I_\rho}\int_0^1 (\psi - w_x)^2 dx + \frac{16\gamma^2}{9I_\rho}\int_0^1 s^2 dx$$
$$\le \frac{G}{4I_\rho}\int_0^1 (\psi - w_x)^2 dx + \frac{16\gamma^2}{9I_\rho}\int_0^1 s_x^2 dx, \quad (3.23)$$
$$-\frac{4\beta}{3I_\rho}\int_0^1 (\psi - w_x)s_t dx \le \frac{G}{4I_\rho}\int_0^1 (\psi - w_x)^2 dx + \frac{16\beta^2}{9I_\rho}\int_0^1 s_t^2 dx,$$
$$-\int_0^1 (3s_t - \psi_t)s_t dx \le \varepsilon_4\int_0^1 (3s_t - \psi_t)^2 dx + \frac{1}{4\varepsilon_4}\int_0^1 s_t^2 dx,$$
$$\frac{\mu}{\rho}\int_0^1 z(x,1)s_x dx \le \frac{\mu^2}{4\rho^2\varepsilon_5}\int_0^1 s_x^2 dx + \varepsilon_5\int_0^1 z^2(x,1)dx, \quad (3.24)$$

for any $\varepsilon_4, \varepsilon_5 > 0$. The assertion of the lemma follows from the estimates (3.23)–(3.24) and (3.22). □

Lemma 7. *If (w,ψ,s,z) is a solution of (2.1), then the functional F_6, defined by*

$$F_6(t) := \tau\int_0^1\int_0^1 e^{-\sigma\tau}z^2(x,\sigma)d\sigma dx$$

satisfies, for $m_1 > 0$ the estimate:

$$\frac{d}{dt}F_6(t) \le -m_1\int_0^1 z^2(x,1)dx - m_1\tau\int_0^1\int_0^1 z^2(x,\sigma)d\sigma dx + \int_0^1 w_t^2 dx. \quad (3.25)$$

Proof. Differentiate F_6 and use the fourth equation in (2.1) and $z(x,0) = w_t$ as follows

$$\frac{d}{dt}F_6(t) = -2\int_0^1\int_0^1 e^{-\tau\sigma}z(x,\sigma)z_\sigma(x,\sigma)d\sigma dx$$
$$= -\int_0^1\int_0^1 \frac{d}{d\sigma}\left[e^{-\tau\sigma}z^2(x,\sigma)\right]d\sigma dx - \tau\int_0^1\int_0^1 e^{-\tau\sigma}z^2(x,\sigma)d\sigma dx$$
$$= -\int_0^1 \left[e^{-\tau}z^2(x,1) - z^2(x,0)\right]dx - \tau\int_0^1\int_0^1 e^{-\tau\sigma}z^2(x,\sigma)d\sigma dx$$
$$= -\int_0^1 e^{-\tau}z^2(x,1)dx + \int_0^1 w_t^2 dx - \tau\int_0^1\int_0^1 e^{-\tau\sigma}z^2(x,\sigma)d\sigma dx.$$

Observe that, $\forall\ \sigma \in (0,1)$, the relation $e^{-\tau} \le e^{-\sigma\tau} \le 1$ holds. Therefore, for some $m_1 = e^{-\tau}$, we arrive at the estimate (3.25). □

4 Exponential stability

In this section, using the lemmas obtained in Section 3, we state and prove our main stability results.

Lemma 8. *Let $N, N_k, k = 1, \dots, 6$, be positive constants. The functional defined by*

$$\mathcal{L}(t) := NE(t) + \sum_{k=1}^{6} N_k F_k(t), \quad N_k > 0, \; k = 1, \dots, 6, \; t \geq 0 \tag{4.1}$$

satisfies the equivalence relation

$$c_1 E(t) \leq \mathcal{L}(t) \leq c_2 E(t), \quad \forall\, t \geq 0, \tag{4.2}$$

for some positive constants c_1 and c_2.

Proof. Let $\mathfrak{L}(t) = \sum_{k=1}^{6} N_k F_k(t)$.

$$\begin{aligned}
|\mathfrak{L}(t)| \leq{}& \rho N_1 \int_0^1 |w w_t| dx + \rho N_1 \int_0^1 \left| w_t \int_0^x \psi(y) dy \right| dx \\
&+ I_\rho N_2 \int_0^1 |(3s_x - \psi_x)(3s_t - \psi_t)|\, dx + 3I_\rho N_3 \int_0^1 |s_t s|\, dx \\
&+ 2\beta N_3 \int_0^1 s^2 dx + 3\rho N_3 \int_0^1 \left| w_t \int_0^x s(y) dy \right| dx \\
&+ 3N_4 \int_0^1 |(3s_t - \psi_t) s|\, dx + N_4 \int_0^1 |(3s_t - \psi_t) w_x|\, dx \\
&+ N_4 \int_0^1 |(3s_x - \psi_x) w_t|\, dx + N_5 \int_0^1 |(\psi - w_x) s_t|\, dx \\
&+ N_5 \int_0^1 |w_t s_x|\, dx + \tau N_6 \int_0^1 \int_0^1 e^{-\sigma\tau} z^2(x, \sigma) d\sigma dx.
\end{aligned}$$

Exploiting Young's, Poincaré's, Cauchy−Schwarz inequalities, (2.2), accompanied with the fact that $w_x = -(\psi - w_x) - (3s - \psi) + 3s$ and $e^{-\sigma\tau} \leq 1$ for all $\sigma \in (0, 1)$, we deduce that for some positive constant η,

$$\begin{aligned}
|\mathfrak{L}(t)| \leq{}& \eta \int_0^1 \left[w_t^2 + (3s_t - \psi_t)^2 + (3s_x - \psi_x)^2 + s_t^2 + s_x^2 + s^2 + (\psi - w_x)^2 \right] dx \\
&+ \eta \int_0^1 \int_0^1 z^2(x, \sigma) d\sigma dx \leq \eta E(t).
\end{aligned}$$

It is easy to observe that, from (4.1) that $|\mathcal{L}(t) - NE(t)| \leq \eta E(t)$, which is equivalent to

$$(N - \eta) E(t) \leq \mathcal{L}(t) \leq (N + \eta) E(t),$$

and hence the relation (4.2) follows by taking N large enough. □

At this point, we're in position to prove our stability result which reads as follows.

Theorem 2. *Let (w, ψ, s, z) be a solution of (2.1) and suppose that $GI_\rho = \rho D$, there exist a positive number $\bar{\mu}$ such that if $|\mu| < \bar{\mu}$, then the energy $E(t)$ of (2.1) defined by (2.2) vanishes exponentially as t approaches infinity, i.e.*

$$E(t) \leq ae^{-bt}, \qquad \forall t \geq 0, \tag{4.3}$$

for some positive constants a and b.

Proof. We proceed by differentiating (4.1), then substitute for functionals F_1 to F_6 using estimates (3.6), (3.9), (3.12), (3.16), (3.20) and (3.25) respectively. Setting

$$N_1 = 1, \quad \varepsilon_1 = \varepsilon_5 = \mu^2, \quad \varepsilon_2 = \frac{\rho}{4N_3}, \quad \varepsilon_3 = \frac{I_\rho N_2}{2N_4}, \;\; N_6 = |\mu|,$$

we end up with

$$\begin{aligned}
\mathcal{L}'(t) \leq &- \left[4\beta N - c_3 - c_3 N_3(1+N_3) - \frac{cN_4^2}{N_2} - c_3 N_5\left(1 + \frac{1}{\varepsilon_4}\right)\right] \int_0^1 s_t^2 dx \\
&- 3\gamma N_3 \int_0^1 s^2 dx - \left[\frac{I_\rho}{2} N_2 - \rho - \varepsilon_4 N_5\right] \int_0^1 (3s_t - \psi_t)^2 dx \\
&- \left[\frac{DN_4}{2I_\rho} - \frac{3DN_2}{2}\right] \int_0^1 (3s_x - \psi_x)^2 dx - [3DN_3 - c_3 N_5] \int_0^1 s_x^2 dx \\
&- \left[\frac{GN_5}{2I_\rho} - c_3 - c_3 N_2 - c_3 N_4\right] \int_0^1 (\psi - w_x)^2 dx \\
&- |\mu| \left[m_1 - |\mu| \left(1 + \frac{9N_3}{4\gamma} + \frac{I_\rho N_4}{D\rho^2} + N_5\right)\right] \int_0^1 z^2(x,1) dx \\
&- m_1 \tau |\mu| \int_0^1 \int_0^1 z^2(x,\sigma) d\sigma dx - \left[\frac{\rho}{4} - |\mu|(N+1)\right] \int_0^1 w_t^2 dx
\end{aligned}$$

for some $c_3 > 0$. Next, we choose N_2 large enough such that

$$k := \frac{I_\rho}{2} N_2 - \rho > 0.$$

Fixing N_2 permits to choose N_4 large enough such that

$$\frac{DN_4}{2I_\rho} - \frac{3DN_2}{2} > 0.$$

With N_2 and N_4 fixed, we can easily choose N_5 large enough such that

$$\frac{GN_5}{2I_\rho} - c_3 - c_3 N_2 - c_3 N_4 > 0.$$

We pick ε_4 adequately small and N_3 sufficiently large such that

$$k - \varepsilon_4 N_5 > 0 \quad \text{and} \quad 3DN_3 - c_3 N_5 > 0$$

respectively. Next, we select N sufficiently large such that (4.2) remains valid and that

$$4\beta N - c_3 - c_3 N_3(1+N_3) - \frac{cN_4^2}{N_2} - c_3 N_5\left(1+\frac{1}{\varepsilon_4}\right) > 0.$$

Finally, pick

$$\bar{\mu} = \min\left\{\frac{m_1}{\left(1+\frac{9N_3}{4\gamma}+\frac{I_\rho N_4}{D\rho^2}+N_5\right)}, \quad \frac{\rho}{4(N+1)}\right\}$$

to we end up with

$$\begin{aligned}\mathcal{L}'(t) \le &-\alpha\int_0^1\left[w_t^2+s_t^2+(3s_t-\psi_t)^2+(3s_x-\psi_x)^2+s_x^2+s^2\right]dx\\ &-\alpha\int_0^1\left[(\psi-w_x)^2+z^2(x,1)+\int_0^1 z^2(x,\sigma)d\sigma\right]dx,\end{aligned}$$

for some $\alpha > 0$. By the virtue of (2.2), it is clear that for some $\alpha_0 > 0$,

$$\mathcal{L}'(t) \le -\alpha_0 E(t), \quad \forall t \ge 0. \tag{4.4}$$

It then follows directly from (4.2) and (4.4) that

$$\mathcal{L}'(t) \le -b\mathcal{L}(t), \quad \forall t \ge 0, \tag{4.5}$$

where $b = \frac{\alpha_0}{c_2}$. A simple integration of (4.5) over $(0,t)$ yields

$$\mathcal{L}(t) \le \mathcal{L}(0)e^{-bt}, \quad \forall t \ge 0. \tag{4.6}$$

Consequently, the assertion of the relation (4.3) follows from (4.6) and (4.2) with $a = c_2 E(0)/c_1$. □

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