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Oscillation Properties for Non-Classical Sturm-Liouville Problems with Additional Transmission Conditions

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Abstract. This work is aimed at studying some comparison and oscillation properties of boundary value problems (BVP's) of a new type, which differ from classical problems in that they are defined on two disjoint intervals and include additional transfer conditions that describe the interaction between the left and right intervals. This type of problems we call boundary value-transmission problems (BVTP's). The main difficulty arises when studying the distribution of zeros of eigenfunctions, since it is unclear how to apply the classical methods of Sturm's theory to problems of this type. We established new criteria for comparison and oscillation properties and new approaches used to obtain these criteria. The obtained results extend and generalizes the Sturm's classical theorems on comparison and oscillation.

Keywords: non-classical SLP's, transmission problems, comparison theorems, oscillatory solutions.

AMS Subject Classification: 34B24; 34L05.

1 Introduction and statement of the problem

The existence and behavior of the zeros of the solutions of Sturm-Liouville equations are of utmost importance in the qualitative theory of ordinary differential equations. Although there are extensive literature on this subject, there is growing interest in deriving a new sufficient conditions for the oscillation or nonoscillation of the solutions of new type non-classical differential equations that have appeared in various fields of natural science in recent years. In particular, non-classical Sturm-Liouville type problems involving additional transmission conditions appear frequently in modeling of different problems encountered in physics such that electromagnetic processes in ferromagnetic media with different dielectric properties, in vibrating folded membranes, in elastic multi-structures, in problems of "hydraulic fracturing" in heat and mass transfer, in vibrations of mechanical systems etc., which, in general, are associated with nonsmooth or discontinuous properties of the medium (see for example, [11, 19, 29, 30]).

The present article devoted to the investigation of comparison and oscillation properties of new type Sturm-Liouville problems involving additional transmission conditions. Our motivation for this work stems from a number of important problems of mathematical physics, in solving of which it is necessary to determine the oscillation properties of solutions. For example, in the study of free oscillations of the Earth, the equation determining the radial dependence of its toroidal oscillations is described by transfer problems for equations of the Sturm-Liouville form(see, [3]). Note that, our problem differs from the classical Sturm-Liouville problems(SLP's) in that it includes additional transmission conditions that describe the relationship between the left and right limits of the solutions and its derivatives at the initial point of interaction. The major difficulty arises in studying the existence and behavior of zeros of solutions and it is required to modify the classical methods such that they can be also applicable to the problems involving additional transmission conditions.

It is known that in studying the various properties of many physical phenomena, it is necessary to obtain sufficient conditions for oscillatory and non-oscillatory solutions, to estimate the minimum distance between consecutive zeros and the maximum number of zeros lying in a given interval, and also to find a connection between such properties of differential equations that arise during modeling such physical phenomena. Therefore the Sturmian comparison and oscillation theory is attracting increasing attention. Sturm's comparison theorem which compares the rapidly of the oscillation of the solutions of the two differential equations, is formulated usually as follows:

Consider the differential equations

$$\frac{d}{dx}\Big(p_1(x)\frac{du}{dx}\Big) - q_1(x)u = 0, \tag{1.1}$$

$$\frac{d}{dx}\Big(p_2(x)\frac{d\upsilon}{dx}\Big) - q_2(x)\upsilon = 0, \tag{1.2}$$

where $p_i(x), q_i(x)(i = 1, 2)$ are continuous functions in an interval $[a, b]$. If $p_1(x) \geq p_2(x) > 0$, $q_1(x) \geq q_2(x) > 0$ for all $x \in [a, b]$ and if x_1, x_2 are two consecutive zeros of a solution $u = u(x)$ of (1.1) then every solution $\upsilon = \upsilon(x)$ of (1.2) has at least one zero in (x_1, x_2).

Note that the Sturm-Liouville equation

$$-\frac{d}{dx}\Big(p(x)\frac{dy}{dx}\Big) + q(x)y = \lambda r(x)$$

is a typical of a large class of equations which arise in problems of many branches of natural science. By the above Sturm's comparison property an increment in $|\lambda|$ produced a more rapid oscillation of the solution. This comparison theorem has some physical interpretations. For example, let a particle is attracted to $u = 0$ by force $f_1(t) = -h_1(t)u$ and suppose that the degree freedom of this particle is equal to one, then the mathematical model of its motion has the Sturm-Liouville form

$$u'' + f_1(t)u = 0,$$

where t stands for time. If another particle attracted to $\nu = 0$ by force $f_2(t) = -h_2(t)\nu$ satisfying $f_2(t) \geq f_1(t)$, $f_2(t) \neq f_1(t)$ (those motion is also modeled by the similar equation $\nu'' + f_2(t)\nu = 0$), then as known from Quantum Theory the second particle will oscillated more rapidly. That is, if the first particle passed through $u = 0$ at successive times t_1 and t_2 then between these moments the second particle (that is a particle with a large force) will pass at least once. Similar investigation for two strings oscillating around an equilibrium line, which is a simple harmonic motion, gives Sturmian comparison and oscillation theorems for hyperbolic equations. This theory can also be utilized to study some qualitative properties of solutions that we cannot find in exact form. Therefore, the Sturmian theory is significant from the theoretical as well as practical point of view. Moreover when studying some problems of mathematical physics, it is required to find oscillatory and non-oscillatory solutions not only for ordinary differential equations (ODE's) but also for partial differential equations (PDE's). The first important results on the comparison and oscillatory properties of solutions of ordinary linear differential equations of second-orders were established by Sturm in 1836 [29]. For classical comparison and oscillation theory for ODE's we refer to Swanson's [30] and for PDE's we refer to Yoshida's [31]. In [19] has been established some important consequences of the Sturm's comparison theorem. There is also a huge literature on applications of this theory (see, for example, [10, 29] and references cited therein). For the further developments and various type generalization we refer to [3, 9, 11, 16, 31, 32] and references cited therein.

In this work we will extend and generalize some well-known Sturm's comparison and oscillation theorems for problems of a new type, which consist of Sturm-Liouville equation

$$\tau y := -y''(x)\} + q(x)y(x) = \lambda y(x), \quad x \in [a, c) \cup (c, b] \tag{1.3}$$

separated boundary conditions

$$\alpha y(a) + \alpha' y'(a) = 0, \tag{1.4}$$

$$\beta y(b) + \beta' y'(b) = 0 \tag{1.5}$$

and additional transmission conditions

$$\delta y(c - 0) + \gamma y(c + 0) = 0, \tag{1.6}$$

$$\delta' y'(c - 0) + \gamma' y'(c + 0) = 0. \tag{1.7}$$

Here we assume that, the real-valued function $q(x)$ is continuous on each of the intervals $[a, c)$ and $(c, b]$ with a finite limits $q(c \pm 0) := \lim_{\epsilon \to 0} q(c \pm |\epsilon|)$; λ is a complex eigenparameter, $\delta, \delta', \gamma, \gamma', \alpha, \alpha', \beta, \beta'$ are real numbers, $|\alpha| + |\beta| \neq 0$, $|\alpha'| + |\beta'| \neq 0$. In recent years, the authors of this study and some other mathematicians have studied many other spectral properties of similar BTVP's (see, for example, [1, 2, 4, 5, 6, 7, 9, 13, 17, 22, 23, 27, 28]). BVTP's also find many important application in the study of various phenomena in physics and technology, such as vibration of string involving different types of loads (see, for example, [12,14,18,21,25,26]), heat transfer through a solid-liquid interface(see, for example, [8,15,20]), water vapor diffusion through a porous membrane (see, for example, [24]).

Remark 1. The considered problem (1.3)–(1.7) differs from the classical SLP's in that it defined on two disjoint intervals and involved additional transfer conditions at one interior point of discontinuity. The main distinguishing feature of such type non-classical problems lies in the behavior of eigenelements. To show this, let us consider the simple, but illustrative BVTP, consisting of two-interval Sturm-Liouville equation

$$y'' + \lambda y = 0, \ x \in [-1, 0) \cup (0, 1] \tag{1.8}$$

with separated boundary conditions

$$y(-1) = y'(1) = 0$$

and with additional transmission conditions at the interior point $x = 0$ given by

$$y(-0) = y(+0), \quad y'(-0) = -y'(+0). \tag{1.9}$$

Although the continuous version of this problem, i.e. the SLP

$$\begin{aligned} &y'' + \lambda y = 0, \quad x \in [-1, 1], \\ &y(-1) = y'(1) = 0 \end{aligned}$$

have infinitely many real eigenvalues $\lambda_n = (\frac{(2n+1)\pi}{2})^2, \quad n = 0, 1, 2, ...,$ the BVTP (1.8)–(1.9) has only trivial solution $y = 0$ for all $\lambda \in \mathbb{R}$, that is this problem has no real eigenvalue.

2 Comparison theorems for two-interval Sturm-Liouville problems

Suppose that $y = y_1(x)$ be any nontrivial solution of the equation

$$y'' + g_1(x)y = 0, \quad x \in [a, c) \cup (c, b]$$

satisfying the transmission conditions

$$\delta y(c - 0) + \gamma y(c + 0) = 0, \quad \delta' y'(c - 0) + \gamma' y'(c + 0) = 0 \tag{2.1}$$

and let $y = y_2(x)$ be any nontrivial solution of the equation

$$y'' + g_2(x)y = 0, \quad x \in [a, c) \cup (c, b]$$

satisfying the same transmission conditions.

Below, for simplicity, we assume that $\delta \neq 0$ and $\delta' \neq 0$.

Theorem 1. *Suppose that $g_1(x) < g_2(x)$ in the whole $[a, c) \cup (c, b]$, and*

$$\frac{\gamma\gamma' - \delta\delta'}{\delta\delta'} W(y_1, y_2; c - 0) < 0,$$

where, as usual $W(y_1, y_2; x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$ denotes the Wronskian's of the solutions y_1, y_2. Then between any two consecutive zeros of the solution $y_1(x)$ there is at least one zero of the solution $y_2(x)$.

Proof. Let x_1 and x_2 be two successive zeros of the solution $y_1(x)$. Since the cases $x_1, x_2 \in (a, c)$ and $x_1, x_2 \in (c, b)$ are totally similar to the classical case one (see, [30]), we will consider only the case $x_1 < c < x_2$. Suppose, if possible, that $y_2(x) \neq 0$ inside $(x_1, c) \cup (c, x_2)$. For definiteness, we put $y_1(x) > 0, \ \ y_2(x) > 0$ over $(x_1, c) \cup (c, x_2)$. Well-known Lagrange's identity (see, [30]) gives

$$\frac{d}{dx} W(y_1, y_2; x) = (g_1(x) - g_2(x))y_1(x)y_2(x). \tag{2.2}$$

Integrating the identity (2.2) and then using transmission conditions (2.1), yields

$$\begin{aligned} &\frac{\gamma\gamma' - \delta\delta'}{\delta\delta'} W(y_1, y_2; c - 0) + y_1'(x_2)y_2(x_2) - y_1'(x_1)y_2(x_1) \\ &= \int_{x_1}^{c-0} (g_2(x) - g_1(x))y_1(x)y_2(x)dx + \int_{c+0}^{x_2} (g_2(x) - g_1(x))y_1(x)y_2(x)dx. \end{aligned} \tag{2.3}$$

Since

$$\lim_{x \to x_1^+} \frac{y_1(x) - y_1(x_1)}{x - x_1} = \lim_{x \to x_1^+} \frac{y_1(x)}{x - x_1} \geq 0,$$

we have that $y_1'(x_1) \geq 0$. Similarly, $y_1'(x_2) \leq 0$. Consequently, the right side of the identity (2.3) is positive, but the left side is negative. Thus we arrive at a contradiction which completes the proof. □

From this comparison criteria there follows the following corollaries, which are important for further investigation.

Corollary 1. Let $y = y(x, \lambda)$ be any solution of the equation (1.3) that satisfies the transmission conditions (1.6)–(1.7). Suppose that $\lambda_1 < \lambda_2$ be any two values of parameter such that

$$\frac{\gamma\gamma' - \delta\delta'}{\delta\delta'} W(y(., \lambda_1), y(., \lambda_2); c - 0) < 0.$$

Then between any two successive zeros of the solution $y(x, \lambda_1)$ has not fewer than one zero of the solution $y(x, \lambda_2)$.

Corollary 2. Let $y = \varphi(x,\lambda)$ and $y = \psi(x,\lambda)$ are two linearly independent solutions of the equation (1.3) that satisfies the transmission conditions (1.6)–(1.7) if

$$\frac{\gamma\gamma' - \delta\delta'}{\delta\delta'} W(\varphi(.,\lambda_0), \psi(.,\lambda_0); c-0) \leq 0$$

for some $\lambda = \lambda_0$, then the zeros of the solutions $\varphi(x,\lambda_0)$ and $\psi(x,\lambda_0)$ separate one another.

Theorem 2. *Let $y = \varphi(x,\lambda)$ be the solution of the equation (1.3) satisfying the initial conditions*

$$\varphi(a,\lambda) = \alpha', \quad \varphi'_x(a,\lambda) = -\alpha \tag{2.4}$$

and the transmission conditions

$$\delta\varphi(c-0,\lambda) + \gamma\varphi(c+0,\lambda) = 0, \quad \delta'\varphi'_x(c-0,\lambda) + \gamma'\varphi'_x(c+0,\lambda) = 0. \tag{2.5}$$

Furthermore, assume that

$$\frac{\gamma\gamma' - \delta\delta'}{\delta\delta'} W(\varphi(x,\lambda_1), \varphi(x,\lambda_2); c-0) < 0$$

for some real λ_1 and λ_2, with $\lambda_1 < \lambda_2$. If $\varphi(x,\lambda_1)$ has m zeros in $(a,c) \cup (c,b]$ then $\varphi(x,\lambda_2)$ has at least m zeros in the same domain $(a,c) \cup (c,b]$ and the i-th zero of $\varphi(x,\lambda_2)$ is less than i-th zero of $\varphi(x,\lambda_1)$.

Proof. Let $x_1, x_2, ..., x_m$ be consecutive zeros of the solution $\varphi(x,\lambda_1)$ lying in $(a,c) \cup (c,b]$. By virtue of the preceding theorem there is at least one zero of $\varphi(x,\lambda_2)$ in each of intervals (x_i, x_{i+1}), $\quad i = 1, 2, ..., n-1$. Therefore it is suffices to show that the solution $\varphi(x,\lambda_2)$ has at least one zero inside (a, x_1).

Case 1. Let $x_1 < c$. Assume the contrary i.e. assume that $\varphi(x,\lambda_2)$ does not vanish inside $[a, x_1]$. Then without loss of generality we may assume that both $\varphi(x,\lambda_1)$ and $\varphi(x,\lambda_2)$ are positive inside $[a, x_1]$. Integrating the identity (2.2) from $x = a$ to $x = x_1$, yields

$$\varphi'_x(x_1,\lambda_1)\varphi(x_1,\lambda_2) = \int_a^{x_1} (\lambda_2 - \lambda_1)\varphi(x,\lambda_1)\varphi(x,\lambda_2)dx. \tag{2.6}$$

Obviously, the right side of this identity is positive. Since $\varphi(x_1,\lambda_1) = 0$ and $\varphi(x,\lambda_1) > 0$ inside $[a, x_1]$, it follows that

$$\varphi'_x(x_1,\lambda_1) = \lim_{x \to x_1} \frac{\varphi(x,\lambda_1)}{x - x_1} = \lim_{x \to x_1^-} \frac{\varphi(x,\lambda_1)}{x - x_1} \geq 0.$$

Therefore the left side of the identity (2.6) is not positive (in fact, the left side of (2.6) is negative, since $\varphi'_x(x_1,\lambda_1)$ can not vanish at $x = x_1$, otherwise then it would follow from the well-know existence and uniqueness theorem from the theory of ordinary differential equations that $\varphi(x,\lambda_1) \equiv 0$ which is impossible). Thus we have a contradiction, which proves the theorem when the first zero

$x = x_1$ of the function $\varphi(x, \lambda_1)$ lying in the left interval (a, c) i.e. for the case $x_1 < c$.

Case 2. Let $x_1 > c$. Assume the contrary, i.e. assume that $\varphi(x, \lambda_2) \neq 0$ in $(a, c) \cup (c, x_1)$. Integrating both sides of the identity (2.2) from a to x_1 and taking in view that the integral is improper at $x = c$ yields

$$W(\varphi(x,\lambda_1), \varphi(x,\lambda_2))|_a^{c-0} + W(\varphi(x,\lambda_1), \varphi(x,\lambda_2))|_{c+0}^{x_1}$$
$$= (\lambda_2 - \lambda_1)(\int_a^{c-0} \varphi(x,\lambda_1)\varphi(x,\lambda_2)dx + \int_{c+0}^{x_1} \varphi(x,\lambda_1), \varphi(x,\lambda_2)dx). \quad (2.7)$$

Using the transmission conditions (2.5) the relation (2.7) is reduced to the following form

$$\frac{\gamma\gamma' - \delta\delta'}{\delta\delta'} W(\varphi(x,\lambda_1), \varphi(x,\lambda_2); c-0) + \varphi_x'(x,\lambda_1), \varphi(x,\lambda_2)$$
$$= (\lambda_2 - \lambda_1)(\int_a^{c-0} \varphi(x,\lambda_1), \varphi(x,\lambda_2)dx + \int_{c+0}^{b} \varphi(x,\lambda_1), \varphi(x,\lambda_2)dx). \quad (2.8)$$

Then both $\varphi(x, \lambda_1)$ and $\varphi(x, \lambda_2)$ does not change its sign in both intervals (a, c) and (c, x_1). For definiteness we put $\varphi(x, \lambda_1) < 0$ and $\varphi(x, \lambda_2) < 0$ in both (a, c) and (c, x_1). These conditions ensure that the right side of (2.8) is positive. On the other hand, since $\varphi(x, \lambda_1) = 0$ and $\varphi(x, \lambda_1) < 0$ for $x \in (c, x_1)$ the function $\varphi(x, \lambda_1)$ is decreasing at the point $x = x_1$, i.e. $\varphi_x'(x_1, \lambda_1) \leq 0$. Therefore the left side of the identity (2.8) is negative. Thus we have a contradiction, which proves that the solution $\varphi(x, \lambda_2)$ has at least one zero in (a, x_1). The proof is complete. □

3 Oscillations properties of the solutions

For further consideration we need the next theorem which assert that the zeros of the solution $\varphi(x, \lambda)$ are continuous functions with respect to the variable λ.

Theorem 3. *Let $x_0 \in (a, c) \cup (c, b)$ be any zero of $\varphi(x, \lambda_0)$. Then for any sufficiently small $\epsilon > 0$ there exists $\delta > 0$ such that for any λ_0' satisfying $|\lambda_0' - \lambda_0| < \delta$ the solution $\varphi(x, \lambda_0')$ has only one zero in the neighborhood $(x_0 - \epsilon, x_0 + \epsilon)$.*

Proof. Let $\varphi_x'(x_0, \lambda_0) > 0$. Since $\varphi_x'(x, \lambda_0)$ is continuous as a function of a variable x, it follows that $\varphi_x'(x, \lambda_0) > 0$ in the interval $(x_0 - \epsilon, x_0 + \epsilon)$ for sufficiently small $\epsilon > 0$. Then $\varphi(x_0 - \epsilon, \lambda_0) < 0$ and $\varphi(x_0 + \epsilon, \lambda_0) > 0$. Furthermore, in view of the continuity of $\varphi_x'(x, \lambda)$ with respect to the variable λ there exists $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ the function $\varphi_x'(x, \lambda)$ also remains positive in whole interval $(x_0 - \epsilon, x_0 + \epsilon)$. On the other hand we can choose $\delta > 0$ so small that $\varphi(x_0 - \epsilon, \lambda) < 0$ and $\varphi(x_0 + \epsilon, \lambda) > 0$, provided that $|\lambda - \lambda_0| < \delta$. Consequently, for $|\lambda - \lambda_0| < \delta$ the function $\varphi(x, \lambda)$ has at least one zero in $(x_0 - \epsilon, x_0 + \epsilon)$. Thus we have proved that the solution $\varphi(x, \lambda)$ for $|\lambda - \lambda_0| < \delta$ has precisely one zero in the interval $(x_0 - \epsilon, x_0 + \epsilon)$. The case $x_0 \in (c, b)$ is totally similar. The proof is complete. □

Remark 2. The obtained results about zeros of the solution $\varphi(x,\lambda)$ can be interpreted as follows. As λ varies, the solution $\varphi(x,\lambda)$ can lose zero or acquire zero only if the zero gets inside or outside the intervals $[a,c)$ and $(c,b]$ through one of the endpoints $x=a$ and $x=b$ or through of the point of interaction $x=c$. In particular, as λ increases, the zeros of the solution $\varphi(x,\lambda)$ move to left, but cannot cross the endpoint $x=a$, since the number of zeros does not decrease and new zeros enter through the right end point.

Now consider the following auxiliary initial-transmission problem

$$-y''+q(x)y=\lambda y, \quad x\in[a,c)\cup(c,b], \tag{3.1}$$

$$y(a)=\alpha', \quad y'(a)=-\alpha,$$

$$\delta y(c^-)+\gamma y(c^+)=0, \quad \delta' y'(c^-)+\gamma' y'(c^+)=0. \tag{3.2}$$

This problem has unique solution $y=\varphi(x,\lambda)$, which is the entire function with respect to the complex variable λ for each $x\in[a,c)\cup(c,b]$(see, [6]).

Lemma 1. *The zeros of the entire function $\varphi(b,\lambda)$ forms an unbounded increasing sequence $\lambda=\mu_n,\ n=0,1,2,...,$ which has the property that the function $\varphi(x,\mu_n)$ has precisely n zeros inside $(a,c)\cup(c,b)$ and the function $\frac{\varphi'(b,\lambda)}{\varphi(b,\lambda)}$ decreases monotonically in each of the intervals (μ_n,μ_{n+1}). Moreover,*

$$\lim_{\lambda\to\mu_n^+}\frac{\varphi'(b,\lambda)}{\varphi(b,\lambda)}=+\infty, \qquad \lim_{\lambda\to\mu_{n+1}^-}\frac{\varphi'(b,\lambda)}{\varphi(b,\lambda)}=-\infty.$$

Proof. Since $q(x)$ is continuous in $[a,c)$ and $(c,b]$ with the finite limits $q(c\mp 0)=\lim_{x\to c\mp 0}q(x)$, there is $T>0$ such that $|q(x)|<T$ in whole $[a,c)\cup(c,b]$. Let us construct the function $\psi(x,\lambda)$ as follows

$$\psi(x,\lambda)=\begin{cases}\alpha'\cosh(\sqrt{-(\lambda+T)}(x-a))\\ -\alpha\sqrt{-(\lambda+T)}\sinh(\sqrt{-(\lambda+T)}(x-a)) \ \ for\ x\in[a,c),\\ \{\frac{-\delta\alpha'}{\gamma}\cosh(\sqrt{-(\lambda+T)}(c-a))\\ +\frac{\delta\alpha}{\gamma}\sqrt{-(\lambda+T)}ch(\sqrt{-(\lambda+T)}(x-c))\}\\ \times\sinh\sqrt{-\lambda-T}(x-a)\\ +\frac{\alpha'\delta'}{\gamma'}\sqrt{-(\lambda+T)}sh(\sqrt{-(\lambda+T)}(x-a))\\ -\frac{\alpha\delta'}{\gamma'}(\lambda+T)ch(\sqrt{-(\lambda+T)}(c-a)), \ \ for\ \ x\ \in(c,b].\end{cases}$$

We can show that this function is the solution of the initial-transmission problem (3.1)–(3.2). Obviously, for negative values λ which are sufficiently large in absolute value, the solution $\psi(x,\lambda)$ does not vanish in $[a,c)\cup(c,b]$. Then by virtue of the Theorem 1 the solution $\varphi(x,\lambda)$ also does not vanish in $[a,c)\cup(c,b]$ for negative values λ with sufficiently large in absolute value. Now we will compare equation (1.3) with the equation

$$-y''-\eta y=-\lambda y, \quad x\in[a,c)\cup(c,b].$$

By using the similar method we can show that the number of the zeros of the solution $\varphi(x,\lambda)$ lying in $[a,c)\cup(c,b]$ increases unboundedly if λ is positive and infinitely increasing. Furthermore, these zeros are continuous functions of λ. By Theorem 2 as λ increases the number of zeros does not decreases, and every zero shifts to the left, but cannot cross the left endpoint $x = a$ and new zeros enter through the right endpoint $x = b$. Therefore, there are infinitely many zeros of the function $\varphi(b, \lambda)$, which are bounded below. Denote by μ_0 the first value of parameter λ for which $\varphi(b, \mu_0) = 0$. Let μ_1 be the second value of the parameter λ for which $\varphi(b, \mu_1) = 0$ and so on. It is evident that for each $n = 0, 1, 2, ...$ the solution $\varphi(x, \mu_n)$ has exactly n zeros inside $[a, c) \cup (c, b]$ and $\varphi(b, \mu_n) = 0$. Now let λ_1 and λ_2 be arbitrary values of parameter λ such that

$$\mu_n < \lambda_1 < \lambda_2 < \mu_{n+1}.$$

It is easy to show that

$$\frac{d}{dx}\left(\frac{W(\varphi(x,\lambda_2),\varphi(x,\lambda_1))\varphi(x,\lambda_1)}{\varphi(x,\lambda_2)}\right) \tag{3.3}$$
$$= \left(\frac{W(\varphi(x,\lambda_2),\varphi(x,\lambda_1))}{\varphi(x,\lambda_2)}\right)^2 + (\lambda_2-\lambda_1)(\varphi(x,\lambda_1))^2 > 0.$$

Therefore, the function $\frac{W(\varphi(x,\lambda_2),\varphi(x,\lambda_1))\varphi(x,\lambda_1)}{\varphi(x,\lambda_2)}$ is monotonically increasing function in any interval, where $\varphi(x, \lambda_2) \neq 0$. Since $\mu_n < \lambda_1 < \lambda_2 < \mu_{n+1}$, both of the solutions $\varphi(x, \lambda_1)$ and $\varphi(x, \lambda_2)$ has exactly n zeros in $[a, c) \cup (c, b]$. Let $x'_1 < x'_2 < ... < x'_n$ are zeros of $\varphi(x, \lambda_1)$ and let $x''_1 < x''_2 < ... < x''_n$ are zeros of $\varphi(x, \lambda_2)$ lying inside $[a, c) \cup (c, b]$. By Theorem 2 $x''_1 < x'_1 < x''_2 < x'_2 < ... < x''_n < x'_n$. We will investigate the cases $x'_n > c$ and $x'_n < c$ separately.

Case 1. Let $x'_n > c$. Then integrating both sides of (3.3) from x'_n to b we obtain

$$\frac{W(\varphi(.,\lambda_2),\varphi(.,\lambda_1);b)\varphi(b,\lambda_1)}{\varphi(b,\lambda_2)} > \frac{W(\varphi(.,\lambda_2),\varphi(.,\lambda_1);x'_n)}{\varphi(x'_n,\lambda_2)}$$
$$+ (\lambda_2-\lambda_1)\int_{x'_n}^{b}\varphi^2(x,\lambda_1)dx = (\lambda_2-\lambda_1)\int_{x'_n}^{b}\varphi^2(x,\lambda_1)dx > 0.$$

From this inequality follows immediately that

$$\frac{\varphi'(b,\lambda_1)}{\varphi(b,\lambda_1)} > \frac{\varphi'(b,\lambda_2)}{\varphi(b,\lambda_2)},$$

so the function $\frac{\varphi'(b,\lambda)}{\varphi(b,\lambda)}$ decreases monotonically in each of intervals (μ_m, μ_{m+1}).

Case 2. Let $x'_n < c$. In this case the integral over $[x'_n, b]$ is improper at the interior point $x = c$. We have to break it into two parts and then use the transmission conditions at $x = c$. Then we get

$$\frac{W(\varphi(.,\lambda_2),\varphi(.,\lambda_1);b)\varphi(b,\lambda_1)}{\varphi(b,\lambda_2)} > W(\varphi(.,\lambda_2),\varphi(.,\lambda_1);c+0)$$
$$- W(\varphi(.,\lambda_2),\varphi(.,\lambda_1);c-0) = \frac{\gamma\gamma'-\delta\delta'}{\delta\delta'}W(\varphi(.,\lambda_2),\varphi(.,\lambda_1);c+0) > 0.$$

From this equality it follows easily that

$$\frac{\varphi'(b,\lambda_1)}{\varphi(b,\lambda_1)} > \frac{\varphi'(b,\lambda_2)}{\varphi(b,\lambda_2)}.$$

Hence, in both cases $x'_n < c$ and $x'_n > c$, the function $\frac{\varphi'(b,\lambda)}{\varphi(b,\lambda)}$ decreases monotonically in each of the intervals (μ_n, μ_{n+1}). Moreover, taking in view the equalities $\varphi(b,\mu_n) = \varphi(b,\mu_{n+1}) = 0$ we see that the function $\frac{\varphi'(b,\lambda)}{\varphi(b,\lambda)}$ should be decrease from $+\infty$ to $-\infty$, which completes the proof. □

Now we are ready to prove the following important oscillation theorem about eigenvalues and eigenfunctions of the main boundary value transmission problem (1.3)–(1.7).

Theorem 4. *There are infinity of real eigenvalues $\lambda_0, \lambda_1, \lambda_2, ...$ of the BVTP (1.3)–(1.7) forming an increasing sequence without a finite limit (i.e. $\lambda_n \to +\infty$ as $n \to \infty$) and the eigenfunction $\varphi(x,\lambda_n)$ corresponding to the eigenvalue λ_n has precisely n zeros in the domain $(a,c) \cup (c,b)$, provided that*

$$\frac{\gamma\gamma' - \delta\delta'}{\delta\delta'} W(\varphi(.,\lambda_{n+1})\varphi(.,\lambda_n); c+0) < 0, \ \textit{for } n = 0, 1, 2, \ldots.$$

Proof. We will consider the cases $\beta' = 0$ and $\beta' \neq 0$ separately.
Case 1. Let $\beta' = 0$. It is evident that for arbitrary λ the function $\varphi(x,\lambda)$ satisfy the equation (1.3), the first boundary condition (1.4) and both transmission conditions (1.6)–(1.7). If $\beta' = 0$, then the second boundary condition (1.5) will be satisfied for $\varphi(x,\mu_n)$, since $\varphi(b,\mu_n) = 0$. Consequently the numbers μ_n are the eigenvalues, i.e. the theorem is proved in this case.
Case 2. Now, let $\beta' \neq 0$. Since the function $\frac{\varphi'(b,\lambda)}{\varphi(b,\lambda)}$ decreases monotonically from $+\infty$ to $-\infty$ in each interval (μ_n, μ_{n+1}), there exists a value $\lambda_n \in (\mu_n, \mu_{n+1})$ for which $\frac{\varphi'(b,\lambda_n)}{\varphi(b,\lambda_n)} = \frac{-\beta}{\beta'}$, i.e. $\beta\varphi(b,\lambda_n) + \beta'\varphi'(b,\lambda_n) = 0$. Consequently the second boundary condition (1.5) holds for $\varphi(x,\lambda_n)$. Thus, λ_n is an eigenvalue with corresponding eigenfunction $\varphi(x,\mu_n)$. Moreover, since $\mu_n < \lambda_n < \mu_{n+1}$, by Theorem 2 the eigenfunction $\varphi(x,\mu_n)$ has as many zeros in $(a,c) \cup (c,b)$ as $\varphi(x,\mu_n)$, i.e. exactly n. The proof is complete. □

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