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THE ROOT CONDITION FOR POLYNOMIAL OF THE SECOND ORDER AND A SPECTRAL STABILITY OF FINITE-DIFFERENCE SCHEMES FOR KURAMOTO-TSUZUKI EQUATION

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ABSTRACT

This paper deals with a root condition for polynomial of the second order. We prove the root criterion for such polynomial with complex coefficients. The criterion coincides with well-known Hurwitz criterion in the case of real coefficients. We apply this root criterion for several three-layer finite-difference schemes for Kuramoto-Tsuzuki equation. We investigate polynomials for symmetrical and DuFort-Frankel finite-difference schemes and polynomial for an odd-even scheme. We establish spectral (conditional or unconditional) stability for these schemes.

A stability concept for discrete problems is of the most importance in the numerical analysis. Since the stability and consistency imply convergence. The von Neumann stability definition is used for problems with constant coefficients. It requires that all eigenvalues of the characteristic equation (or the amplification matrix) be in the closed unit disc and the ones on the unit circle be simple [13]. For finite-difference schemes we can get necessary stability conditions from a spectral (von Neumann) stability analysis [1]. In particular, von Neumann's condition is necessary for stability in L_2. Often these necessary conditions are sufficient conditions for linear finite-difference schemes too. The definition of spectral stability appears when we investigate stability of numerical integration methods (Runge-Kutta, multistep methods) for ordinary differential equations [1; 5; 6; 7; 16] and partial differential equations [1; 15]. Thus, we built characteristic equations for various discrete problems and investigate all roots of this equation (polynomial).

1. ROOT CONDITION

Consider a complex polynomial

$$f(z) = P_m(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0 \qquad (1.1)$$

with coefficients $a_i \in \mathbf{C}$ where $\mathbf{C}$ is a set of complex numbers. If $a_m \neq 0$ then such polynomial has m roots $q_i \in \mathbf{C}, i = 1, \ldots, m$ exactly.

Now we formulate *the root condition* [1; 16] for polynomial (1.1) (see Fig. 1).

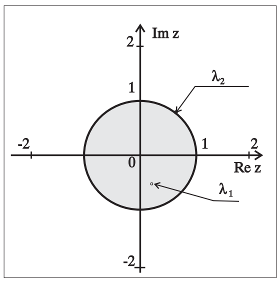


Figure 1. Root condition.

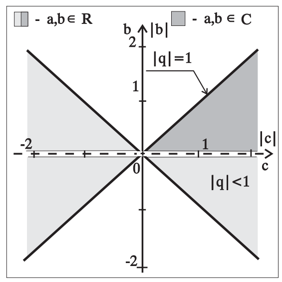


Figure 2. Criterion for linear polynomial($b \neq 0$).

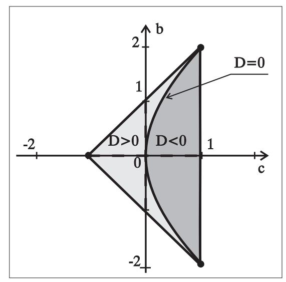


Figure 3. Hurwitz's criterion.

DEFINITION 1.1. Polynomial P_m satisfies *the root condition* if all roots of this polynomial are in the closed unit disc of complex plane and those roots of magnitude 1 are simple.

Usually we use two-level or three-level finite-difference schemes for nonstationary partial differential equations [1; 15; 16] . In this case we get linear polynomial $P_1(z) = bz + c$ or the second order polynomial $P_2(z) = az^2 + bz + c$. There are no problems with the linear equation because it has only one root q (if $b \neq 0$). We must check the magnitude of this root. If it is less or equal than 1 then polynomial satisfies the root condition. Thus, in this case we have equality

$$A \equiv \left\{(c, b) \in \mathbf{C}^2, |q| \leq 1\right\} = \left\{|c| \leq |b|, \, b \neq 0\right\} \qquad (1.2)$$

and an obvious criterion for the linear polynomial. Therefore, to check the root condition for this polynomial, we simply check whether both coefficients b and c belong to the root condition set A (see Fig. 2 with axes $|c|$ and $|b|$ in the general case and axes c and b in the real polynomial case). If $b = 0$ and $c \neq 0$ then there are no roots. If $b = 0$ and $c = 0$ then all $q \in \mathbf{C}$ are roots of linear equation, i.e. linear polynomial does not satisfy the root condition.

In general case we have no such a simple criterion for polynomial (1.1). Let us denote by p ($p \leq n$) the number of zeros of the polynomial $f(z)$ which are in the unit circle $|z| = 1$. One of the ways to determine p is to map the interior

$|z| < 1$ of the unit circle into the left half of the complex plane $\operatorname{Re} z < 0$. Then the number of zeros may be found by using Hurwitz's criterion [4; 8; 14] for a new polynomial $F(z) = z^n + A_1 z^{n-1} + \ldots + A_n$ in this domain.

THEOREM 1.2. [Hurwitz's criterion] *If all the determinants*

$$\begin{vmatrix} A_1 & A_3 & A_5 & \ldots & A_{2k-1} \\ 1 & A_2 & A_4 & \ldots & A_{2k-2} \\ 0 & A_1 & A_3 & \ldots & A_{2k-3} \\ 0 & 1 & A_2 & \ldots & A_{2k-4} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & A_k \end{vmatrix}$$

for $k = 2, \ldots, n$ with $A_j = 0$ for $j > n$ are positive, then the polynomial $F(z)$ has zeros only with negative real parts.

A different way to determine p is to use theorems for unit circle, such as *Schur-Cohn* criterion. Consider the polynomial (1.1).

THEOREM 1.3. [Schur-Cohn criterion] *If for polynomial $f(z)$ all the determinants*

$$\Delta_k = \begin{vmatrix} a_0 & 0 & 0 & \ldots & 0 & a_n & a_{n-1} & \ldots & a_{n-k+1} \\ a_1 & a_0 & 0 & \ldots & 0 & 0 & a_n & \ldots & a_{n-k+2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{k-1} & a_{k-2} & a_{k-3} & \ldots & a_0 & 0 & 0 & \ldots & a_n \\ \bar{a}_n & 0 & 0 & \ldots & 0 & \bar{a}_0 & \bar{a}_1 & \ldots & \bar{a}_{k-1} \\ \bar{a}_{n-1} & \bar{a}_n & 0 & \ldots & 0 & 0 & \bar{a}_0 & \ldots & \bar{a}_{k-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \bar{a}_{n-k+1} & \bar{a}_{n-k+2} & \bar{a}_{n-k+3} & \ldots & \bar{a}_n & 0 & 0 & \ldots & \bar{a}_0 \end{vmatrix}$$

are different from zero, then $f(z)$ has no zeros on the circle $|z| = 1$ and it has p zeros inside this circle, p being the numbers of variations of sign in the sequence $1, \Delta_1, \ldots, \Delta_n$.

This criterion is due to Schur [17; 18] in the case $\Delta_k > 0$ for all k and essentially to Cohn [3] in the general case.

Let us associate with $f(z)$ the polynomial

$$f^*(z) = z^n \bar{f}(1/z) = \bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_n = \bar{a}_0 \prod_{j=1}^{n} (z - z_j^*)$$

whose zeros $z_k^* = 1/\bar{z}_k$ are, relative to circle $|z| = 1$, the inverses of the zeros z_k of $f(z)$.

Now we shall follow Morris Marden's "Geometry of Polynomials" [11]. For $f(z)$ and $f^*(z)$ we construct the sequence of polynomials $f_j(z) = \sum_{k=0}^{n-j} a_k^{(j)} z^k$, where $f_0(z) = f(z)$ and

$$f_{j+1}(z) = \bar{a}_0^{(j)} f_j(z) - a_{n-j}^{(j)} f_j^*(z), \quad j = 0, 1, \ldots, n-1.$$

Thus,

$$a_k^{(j+1)} = \bar{a}_0^{(j)} a_k^{(j)} - a_{n-j}^{(j)} \bar{a}_{n-j-k}^{(j)}.$$

The constant term $a_0^{(j)}$ in each polynomial $f_j(z)$ is a real number which we denote by δ_j :

$$\delta_{j+1} = |a_0^{(j)}|^2 - |a_{n-j}^{(j)}|^2 = a_0^{(j+1)}, \quad j = 0, 1, 2, \ldots, n-1.$$

As to the zeros of these polynomials, Cohn [3] has proved lemma which we present in the compact form due to Marden [10]:

LEMMA 1.4. *If f_j has p_j zeros interior to the unit circle $C : |z| = 1$ and if $\delta_{j+1} \neq 0$, then f_{j+1} has*

$$p_{j+1} = (1/2)\Big(n - j - \big((n-j) - 2p_j\big)\operatorname{sign}\delta_{j+1}\Big)$$

zeros interior to C. Furthermore, f_{j+1} has the same zeros on C as f_j.

Marden [10; 11] has proved the following theorem:

THEOREM 1.5. *For a given polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, let the sequence $f_{j+1}(z)$ be constructed. Then, if for some $k < n, P_k \neq 0$ in $P_k = \delta_1 \delta_2 \ldots \delta_k$, $k = 1, 2, \ldots, n$, but $f_{k+1}(z) \equiv 0$, then f has $n-k$ zeros on or symmetric in the circle $C : |z| = 1$ at the zeros of $f_k(z)$. If p of the $P_j, j = 1, 2, \ldots, k$, are negative, then f has p additional zeros inside C and $q = k - p$ additional zeros outside C.*

Returning back to the polynomials $f(z)$ which do not have any zeros on the circle $|z| = 1$, let us consider the case that, $\delta_1 \delta_2 \cdots \delta_k \neq 0$ for some $k < n$, but

$$\delta_{k+1} = a_0^{(k+1)} = |a_0^{(k)}|^2 - |a_{n-k}^{(k)}|^2 = 0.$$

In such a case the number p of zeros of $f(z)$ in the unit circle $C : |z| < 1$ may be found either by a limiting process or by modification of the sequence.

The limiting process may be chosen as one operating upon the circle C or upon the coefficients of $f_k(z)$. That is, since $f_k(z)$ has no zeros on the circle C, we may consider in place of $f_k(z)$ the polynomial

$$F_k(z) = f_k(rz)$$

which, for $r = 1 \pm \varepsilon$ and ε is sufficiently small positive number, has as many zeros in the circle $|z| < 1$ as $f_k(z)$ does.

A more direct procedure to cover the case of a vanishing δ_{k+1} is to modify the sequence. The modification applies even when $f_k(z)$ has zeros for $|z| = 1$.

Observing that the first and the last coefficients in

$$f_k(z) = a_0^{(k)} + a_1^{(k)} z + \cdots + a_{n-k}^{(k)} z^{n-k}$$

are of the same magnitude, we find to be useful the following two theorems due to Cohn [3].

THEOREM 1.6. *If the coefficients of the polynomial* $g(z) = b_0 + b_1 z + \cdots + b_m z^m$ *satisfy the relations:*

$$b_m = u\bar{b}_0, \quad b_{m-1} = u\bar{b}_1, \ldots, b_{m-q+1} = u\bar{b}_{q-1},\ b_{m-q} \neq u\bar{b}_q$$

where $q \leq m/2$ *and* $|u| = 1$, *then* $g(z)$ *has* $|z| \leq 1$ *as many zeros as the polynomial*

$$G_1(z) = \bar{B}_0 G(z) - B_{m+q} G^*(z) = \sum_{j=0}^{m} B_j^{(1)} z^j,$$

where $G(z) = (z^q + 2b/|b|)g(z) = \sum_{j=0}^{m+q} B_j z^j$, $\quad b = (b_{m-q} - u\bar{b}_q)/b_m$, *and* $|B_0^{(1)}| < |B_m^{(1)}|$.

THEOREM 1.7. *If* $g(z) = b_0 + b_1 z + \cdots + b_m z^m$ *is a self-inversive polynomial, i.e. if*

$$b_m = u\bar{b}_0, \quad b_{m-1} = u\bar{b}_1, \ldots, b_0 = u\bar{b}_m,\ |u| = 1,$$

then g *has as many zeros on the disk* $|z| < 1$ *as the polynomial*

$$g_1(z) = [g'(z)]^* = \sum_{j=0}^{m-1} (m-j)\bar{b}_{m-j} z^j$$

has. That is, g *and* g' *have the same number of zeros for* $|z| > 1$.

2. HURWITZ'S CRITERION

We consider only the polynomials of the second order

$$az^2 + bz + c = 0,\ a, b, c \in \mathbf{C},\ a \neq 0. \tag{2.1}$$

The case $a = 0$ deals with linear polynomial. If $a \neq 0$ then we put equation (2.1) into the form with coefficient $\tilde{a} = 1$, $\tilde{b} = b/a$, $\tilde{c} = c/a$.

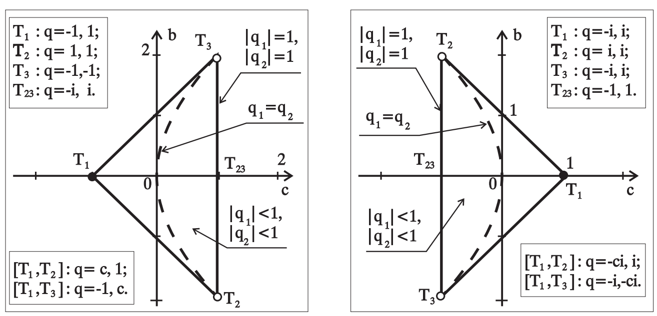
If all the coefficients of the polynomial (2.1) are real numbers ($a = 1$, $b \in \mathbf{R}$, $c \in \mathbf{R}$) then the well-known *Hurwitz's criterion* holds: all (two) roots of real polynomial of the second order are in the unit disk if it's coefficients satisfy the two inequalities

$$|c| \leq 1, \quad |b| \leq c + 1. \tag{2.2}$$

In this case the analysis of the spectral stability is not complicated because we need to verify two simple inequalities. The set of the points (c, b) satisfying Hurwitz's criterion make a triangle (see Fig. 3 and Fig. 4a). We notice that the double root is on the unit circle, when $c = 1$ and $D = b^2 - 4c = 0$, i.e. $|b| = 2$. Then the root condition for real polynomial of the second order becomes

$$A \equiv \big\{ |c| \leq 1, \ |b| \leq c + 1; \quad |b| < 2 \text{ if } \ c = 1 \big\}. \tag{2.3}$$

The inequalities (2.2) and (2.3) for coefficients of the polynomial are simple enough and we can verify easily them for specific finite-difference schemes even if coefficients $b, c \in \mathbf{R}$ depending on different parameters of schemes $(h, \tau, \sigma, \text{etc.})$. In this paper we formulate conditions witch generalize Hurwitz's criterion (root condition) for the polynomial (2.1) with complex coefficients.



$B = b \in \mathbf{R}$ $\qquad$ $B = b\mathbf{i}, \ b \in \mathbf{R}$

Figure 4. Roots in the case $B = b$ and $B = b\mathbf{i}$, $b \in \mathbf{R}$, $c \in \mathbf{R}$.

3. THE ROOT CONDITION FOR COMPLEX POLYNOMIAL OF THE SECOND ORDER

Let the roots of the second order polynomial (2.1) be q_1 and q_2. We denote the set of coefficients of this polynomial $\widetilde{\mathbf{C}} = \big\{(a, b, c) \in \mathbf{C}^3, \ a \neq 0\big\}$ and separate some subsets of this set in the following way:

$$\begin{aligned}
A_0 &= \big\{(a, b, c) \in \widetilde{\mathbf{C}}, \quad |q_1| < 1, \ |q_2| < 1\big\}, \\
A_1 &= \big\{(a, b, c) \in \widetilde{\mathbf{C}}, \quad |q_1| < 1, \ |q_2| = 1\big\}, \\
A_2 &= \big\{(a, b, c) \in \widetilde{\mathbf{C}}, \quad |q_1| = |q_2| = 1, \ q_1 \neq q_2\big\}.
\end{aligned}$$

The root condition holds if the coefficients of the polynomial belong to one of these sets. Then a set $A = A_0 \cup A_1 \cup A_2$ is the root condition set. The analysis of these sets using theorems for zeros in the unit circle from the section 1(theorems 1.5, 1.6,1.7 and see [11]) implies:

$$\begin{aligned} A_0 &= \big\{|c|^2 + |\bar{a}b - \bar{b}c| < |a|^2\big\}, \\ A_1 &= \big\{|c|^2 + |\bar{a}b - \bar{b}c| = |a|^2,\ |c| < |a|\big\}, \\ A_2 &= \big\{|c| = |a|,\ \bar{a}b = \bar{b}c,\ |b| < 2|a|\big\}. \end{aligned}$$

This result we formulate as the following theorem.

THEOREM 3.1. [The root condition] *The roots of the second order polynomial are in the closed unit disc of complex plane and those roots of magnitude 1 are simple if*

$$A = \big\{|c|^2 + |\bar{a}b - \bar{b}c| \le |a|^2,\ |b| < 2|a|\big\}. \tag{3.1}$$

COROLLARY 3.2. *If $a = 1$ then the root condition reduces to*

$$A = \big\{|c|^2 + |b - \bar{b}c| \le 1, \quad |b| < 2\big\}. \tag{3.2}$$

Remark 3.3. If $a = 1, b = \bar{b} \in \mathbf{R}$, $c = \bar{c} \in \mathbf{R}$ then equality (3.2) corresponds to root condition (2.3) in Hurwitz's criterion.

Proof. From (3.2) we get $|c| \le 1$. If $c = 1$ then $|b| < 2$. If $-1 \le c < 1$ then

$$|b|(1-c) = |b - bc| = |b - \bar{b}c| \le 1 - c^2 = (1+c)(1-c),$$

and we get the similar condition to that in Hurwitz's criterion: $|b| \le 1 + c$. □

In applied problems' it is convenient to separate conditions $|c| = |a|$, $|c| < |a|$ and to use the root conditions in the form $A = (A_0 \cup A_1) \cup A_2$:

$$A = \big\{|c| < |a|,\ |c|^2 + |\bar{a}b - \bar{b}c| \le |a|^2\big\} \cup \big\{|c| = |a|,\ \bar{a}b = \bar{b}c,\ |b| < 2|a|\big\}. \tag{3.3}$$

Remark 3.4. If we are interested in problem where double root may appear on the unit circle then in the expressions (3.1), (3.3) strict inequality $|b| < 2|a|$ must be changed to the inequality $|b| \le 2|a|$, because the double root ($q_1 = q_2$) lies on the circle in the case coefficients belong to thee set

$$A_{11} = \big\{|c| = 1,\ \bar{a}b = \bar{b}c,\ |b| = 2|a|\big\}. \tag{3.4}$$

The root criterion for circle $|z| = R \ne 1$ becomes

$$A = \big\{|c|^2 + R|R^2\bar{a}b - \bar{b}c| \le R^4|a|^2,\ |b| < 2R|a|\big\} \tag{3.5}$$

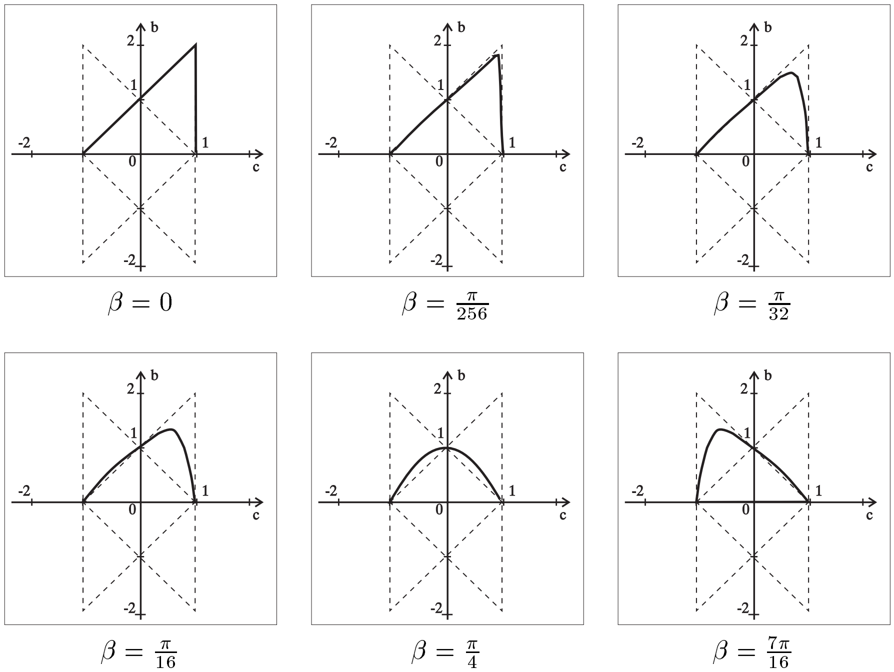


Figure 5. Functions $b = \Phi_\beta(c)$, $0 \le \beta \le \frac{\pi}{2}$.

or in the case $a = 1$

$$A = \big\{|c|^2 + R|R^2 b - \bar{b}c| \le R^4, \quad |b| < 2R\big\}. \tag{3.6}$$

Now we consider equation (2.1) when $a = 1$ in the form

$$w^2 + Bw + C = 0 \tag{3.7}$$

with $B = be^{\beta\mathbf{i}}$, $C = ce^{\beta\mathbf{i}}$, $\beta, \gamma \in [0, 2\pi)$, $b, c \in \mathbf{R}$. Such form of the complex number $Z = ze^{\varphi\mathbf{i}}$, $z \in \mathbf{R}$, $\varphi \in [0, \pi)$ with negative z is equivalent to standard exponential form of complex number $Z = |Z|e^{(\varphi+\pi)\mathbf{i}}$. Thus, every complex number $Z = |Z|e^{\varphi\mathbf{i}}$, $\varphi \in [0, 2\pi)$ we may write in such form.

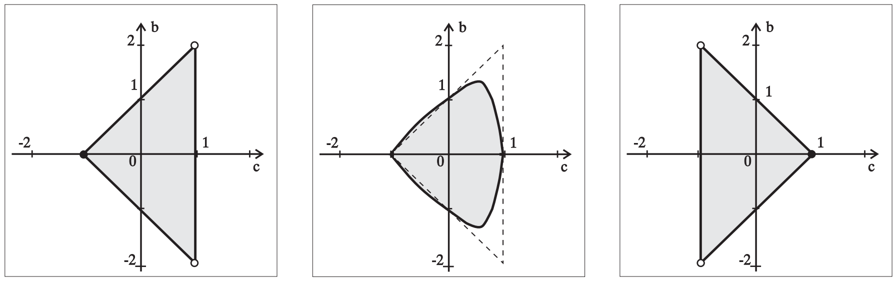
If we put $w = ze^{\frac{1}{2}\gamma\mathbf{i}}$ then equation (3.7) becomes

$$z^2 + be^{(\beta - \frac{1}{2}\gamma)\mathbf{i}} z + c = 0 \tag{3.8}$$

which satisfies the root condition together with polynomial (3.7), because $|z| = |w|$. Thus, we may consider only the case $\gamma = 0$ not loosing generality.

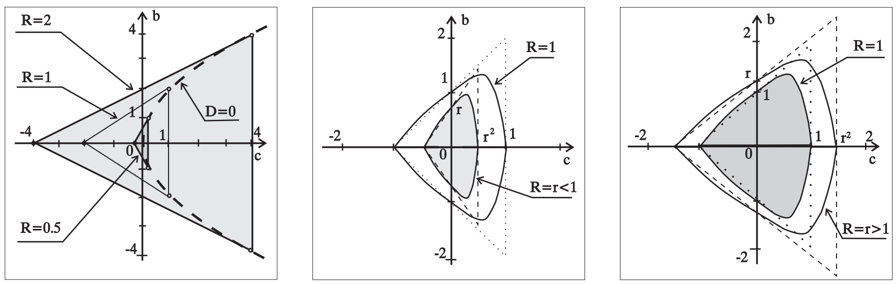
If $\beta = \frac{\pi}{2}$, $\gamma = 0$, i.e. B=b$\mathbf{i}$, then the root condition is

$$A = \big\{|c| < 1, |b\mathbf{i} + b\mathbf{i}c| \le 1 - |c|^2\big\} \cup \big\{|c| = 1, b\mathbf{i} = -b\mathbf{i}c, |b| < 2\big\}$$



$\beta = 0$ $\quad 0 < \beta < \frac{\pi}{2},\ \frac{\pi}{2} < \beta < \pi$ $\quad \beta = \frac{\pi}{2}$

Figure 6. Criterion for $C = c$ and $B = be^{\beta\mathbf{i}}$ with $b, c \in \mathbf{R}$, $0 \le \beta < \pi$.



$\beta = 0$ $\quad R < 1$ $\quad R > 1$

Figure 7. Criterion for $R \neq 1$.

or

$$A = \big\{|c| < 1, |b| \le 1 - c\big\} \cup \big\{c = 1, b = 0\big\} \cup \big\{c = -1, |b| < 2\big\}. \tag{3.9}$$

Thus, in this case the points (c, b) make a triangle (see Fig. 4b).

More complicated case we have for $\beta \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$, $\gamma = 0$. There the condition $B = \overline{B}C$ is equivalent to $be^{\beta\mathbf{i}} = cbe^{-\beta\mathbf{i}}$, i.e. $b = 0$ for $|c| = 1$. For $|c| < 1$ we get $|b| \cdot |e^{\beta\mathbf{i}} - e^{-\beta\mathbf{i}}| \le 1 - |c|^2$ or

$$|b| \le \Phi_\beta(c) = \frac{1 - |c|^2}{\sqrt{1 + |c|^2 - 2c\cos(2\beta)}}. \tag{3.10}$$

Functions $\Phi_\beta(c)$ for various β are sophisticated (see Fig. 5). Thus, the points (c, b) determine different domains for $\beta = 0$, $\beta = \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$, $\frac{\pi}{2} < \beta < \pi$ (see Fig. 6).

To draw the root criterion for $R \neq 1$ in the plane (c, b) we must stretch the domain of root condition R times along b-axis and R^2 times along c-axis (see Fig. 7).

4. THE ROOT CONDITION AND THREE-LAYER FINITE-DIFFERENCE SCHEMES FOR KURAMOTO-TSUZUKI EQUATIONS

We investigate spectral stability for the tree-layer finite-difference schemes for Kuramoto-Tsuzuki equation [9; 12; 19]

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad \kappa = \alpha + \beta \mathbf{i} \neq 0, \ \alpha, \beta \in \mathbf{R}. \tag{4.1}$$

a) *Symmetrical finite-difference scheme* [15]:

$$\frac{\hat{y} - \check{y}}{2\tau} = \kappa \Lambda(\sigma \hat{y} + (1 - 2\sigma) y + \sigma \check{y}), \ \Lambda y_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}. \tag{4.2}$$

By spectral method for this scheme we get the polynomial

$$P(q) = (1 + 2\tau\lambda\kappa\sigma) q^2 + 2\tau\lambda\kappa(1 - 2\sigma) q + (2\tau\lambda\kappa\sigma - 1)$$

with $\lambda = \frac{4}{h^2} \sin^2(\varphi/2)$. First of all we build expressions

$$|a|^2 - |c|^2 = 8\tau\lambda \mathrm{Re}\,(\kappa\sigma), \tag{4.3}$$
$$\bar{a}b - \bar{b}c = 4\tau\lambda(\mathrm{Re}\,\kappa - 2\mathrm{Re}\,(\kappa\sigma) - \tau\lambda|\kappa|^2 \mathrm{Im}\,\sigma\ \mathbf{i}). \tag{4.4}$$

If $\mathrm{Re}\,(\kappa\sigma) = 0$ then $|a| = |c|$ and the root condition (3.3) implies the necessary condition of the stability $\bar{a}b - \bar{b}c = 0$. Equality (4.4) shows that $\mathrm{Re}\,\kappa = 0$ and $\mathrm{Im}\,\sigma = 0$ in this case. Thus, spectrally stable is the difference scheme for only Schrödinger equation and only in the case of real σ. Condition $|b| < 2|a|$ is equivalent to

$$|2\tau\lambda\beta(1 - 2\sigma)| < 2\sqrt{1 + (2\tau\lambda\beta\sigma)^2}$$

which we put to the form

$$\sigma > \frac{1}{4} - \frac{1}{4} \frac{1}{\lambda^2 \tau^2 \beta^2}.$$

Hence, for $\sigma \geq \frac{1}{4}$ the symmetrical finite-difference scheme is unconditionally stable. If $\sigma < \frac{1}{4}$ then we have conditional stability with the condition

$$\frac{\tau}{h^2} < \frac{1}{4|\beta|} \frac{1}{\sqrt{1 - 4\sigma}}. \tag{4.5}$$

Remark 4.1. It is possible to establish non-strict inequality for the first type problem.

If $\mathrm{Re}\,(\kappa\sigma) > 0$ then $|c| < |a|$ and necessary condition of the stability becomes

$$|\mathrm{Re}\,\kappa - \tau\lambda|\kappa|^2 \mathrm{Im}\,\sigma\ \mathbf{i} - 2\mathrm{Re}\,(\kappa\sigma)| \leq 2\mathrm{Re}\,(\kappa\sigma).$$

For all real numbers x, y, a $(a > 0)$ the inequality $|x - y\mathbf{i} - a| \leq a$ is equivalent to $y^2 \leq 2xa - x^2$. Thus, we have condition

$$\tau^2\lambda^2|\kappa|^4(\mathrm{Im}\,\sigma)^2 \leq 4\mathrm{Re}\,(\kappa\sigma)\cdot\mathrm{Re}\,\kappa - (\mathrm{Re}\,\kappa)^2. \tag{4.6}$$

Finite-difference scheme (4.2) isn't stable when $\mathrm{Re}\,\kappa < 0$. If $\mathrm{Re}\,\kappa = \beta = 0$ then from (4.6) we get the condition $\mathrm{Im}\,\sigma = 0$ which isn't compatible with condition $\mathrm{Re}\,(\kappa\sigma) = \mathrm{Re}\,(\beta\sigma\mathbf{i}) > 0$. If $\mathrm{Re}\,\kappa > 0$ then we get the following necessary condition of stability $4\mathrm{Re}\,(\kappa\sigma) \geq \mathrm{Re}\,\kappa > 0$. In this case, if

$$\frac{\tau}{h^2} \leq \frac{\sqrt{4\mathrm{Re}\,(\kappa\sigma) - (\mathrm{Re}\,\kappa)^2}}{4|\kappa|^2|\mathrm{Im}\,\sigma|}, \tag{4.7}$$

then we have conditional stability which passes to the unconditional stability in the case $\mathrm{Im}\,\sigma = 0$.

b) *DuFort-Frankel* finite-difference scheme for Kuramoto-Tsuzuki equation (4.1) is

$$\frac{\hat{y} - \check{y}}{2\tau} + \kappa\frac{\tau^2}{h^2}\ \frac{\hat{y} - 2y + \check{y}}{\tau^2} = \kappa\Lambda y. \tag{4.8}$$

For this scheme we get the second order polynomial

$$P(q) = (1 + \gamma\kappa)q^2 - 2\kappa\gamma\eta q + (\gamma\kappa - 1),$$

with $\gamma = \frac{2\tau}{h^2}, \eta = \cos\varphi$. Then we build expressions

$$\begin{aligned} |a|^2 - |c|^2 &= 4\gamma\mathrm{Re}\,\kappa, \\ \bar{a}b - \bar{b}c &= 4\gamma\eta\mathrm{Re}\,\kappa. \end{aligned}$$

In the case of Schrödinger equation, $\mathrm{Re}\,\kappa = \alpha = 0$ and the condition of stability becomes

$$|2\gamma\eta\beta| < 2|1 + \gamma\beta\ \mathbf{i}|$$

or $\eta^2\gamma^2\beta^2 < 1 + \beta^2\gamma^2$. It holds for all γ and a.

If $\mathrm{Re}\,\kappa > 0$ then the condition of stability becomes $|4\gamma\eta\mathrm{Re}\,\kappa| \leq 4\gamma\mathrm{Re}\,\kappa$ which is true always.

COROLLARY 4.2. *DuFort-Frankel finite-difference scheme is unconditionally stable when* $\mathrm{Re}\,\kappa \geq 0$.

c) *Odd-even finite-difference scheme.* Čiegis and Štikonienė had investigated spectral stability of the odd-even finite-difference scheme [2] for $\kappa = \alpha$ (parabolic equation), $\kappa = \mathbf{i}$ (Schrödinger equation). In the general Kuramoto-Tsuzuki case for such odd-even scheme we have the polynomial

$$P(q) = (1+\kappa\gamma)q^2 - (2-\kappa^2\gamma^2\eta^2)q + (1-\kappa\gamma),$$

with $\gamma = \frac{4\tau}{h^2}$, $|\eta| \le 1$. We apply the proposed method for second order polynomial:

$$\begin{aligned} |a|^2 - |c|^2 &= 4\gamma \mathrm{Re}\,\kappa, && (4.9)\\ \bar{a}b - \bar{b}c &= -4\gamma\mathrm{Re}\,\kappa + 2\eta^2\mathrm{Re}\,\kappa|\kappa|^2 + 4\gamma^2\eta^2\mathrm{Re}\,\kappa\ \mathrm{Im}\,\kappa\ \mathbf{i}. && (4.10)\end{aligned}$$

For this polynomial the equality $|a| = |c|$ holds only for Schrödinger equation when $\mathrm{Re}\,\kappa = 0$. Then from equation (4.10) we get $\bar{a}b = \bar{b}c$ and thee condition of spectral stability becomes

$$2 + \beta^2\gamma^2\eta^2 < 2\sqrt{1+\beta^2\gamma^2}$$

witch is equivalent to $4\eta^2 + \beta^2\gamma^2\eta^4 < 4$ because $\beta^2 > 0$. Finally, we have

$$\gamma\eta^2 < 2\sqrt{1-\eta^2}/\beta. \tag{4.11}$$

In the general case η may be equal to 1. Thus, the odd-even scheme is unstable. Sometimes [2] it is possible to distinguish the cases when $\eta \le m(h) < 1$ and the conditional stability occurs if

$$\frac{4\tau}{h^2} = \gamma < \frac{2\sqrt{1-m^2}}{m^2}.$$

If $\mathrm{Re}\,\kappa > 0$ then $|c| < |a|$ and we have the second case of root condition (3.3). Since

$$|\bar{a}b - \bar{b}c| = 2\gamma\mathrm{Re}\,\kappa\sqrt{(2-\gamma^2\eta^2|\kappa|^2)^2 + 4\gamma^2\eta^4(\mathrm{Im}\,\kappa)^2},$$

the root condition $|\bar{a}b - \bar{b}c| \le |a|^2 - |c|^2$ is equivalent to the inequality

$$2\gamma\mathrm{Re}\,\kappa\sqrt{4 - 4\gamma^2\eta^2|\kappa|^2 + \gamma^4\eta^4|\kappa|^4 + 4\gamma^2\eta^4(\mathrm{Im}\,\kappa)^2} \le 4\gamma\mathrm{Re}\,\kappa$$

or

$$\gamma^2\eta^2|\kappa|^4 \le 4(1-\eta^2)|\kappa|^2 + 4\eta^2(\mathrm{Re}\,\kappa)^2.$$

We notice that $4(1-\eta^2)|\kappa|^2 \ge 0$, therefore the necessary condition of stability becomes

$$\frac{4\tau}{h^2} = \gamma \le \frac{2\mathrm{Re}\,\kappa}{|\kappa|^2}. \tag{4.12}$$

COROLLARY 4.3. *Odd-even finite-difference scheme is conditionally stable when* $\mathrm{Re}\,\kappa > 0$.

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